Nonchaotic and chaotic Behavior in Three-Dimensional Quadratic Systems: Five-One Conservative Cases

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Abstract

In this paper we study the nonchaotic and chaotic behavior of all 3D conservative quadratic ODE systems with five terms on the right-hand side and one nonlinear term (5-1 systems). We prove a theorem which provides sufficient conditions for solutions in 3D autonomous systems being nonchaotic. We show that all but five of these systems:(3.8a,b), (3.11b), (3.34)($A = \mp 1$), (4.1b),and (4.7a,b) are nonchaotic. Numerical simulations show that only one of the five systems, (4.1b), really appears to be chaotic. If proved to be true, it will be the simplest ODE system having chaos.

Keywords: Nonchaotic behavior, quadratic, conservative system, and chaos

1 Introduction

It is well known that three-dimensional quadratic autonomous systems are the simplest type of ordinary differential equations in which it is possible to exhibit chaotic behavior. Lorenz equations (Lorenz, 1963) and Rössler system (Rössler, 1976) both with seven terms on the right-hand side do exhibit chaos for certain parameter values. Very interesting investigations on three-dimensional quadratic systems with less than seven terms and more than four terms on the right-hand side have been carried out by J. C. Sprott (1994, 1997 and 2003). By computer simulation, Sprott found numerous cases of chaos in systems with six terms on the right-hand side with only one nonlinear (quadratic) term and numerous examples of chaotic five-term systems with two nonlinear terms. In a follow-up study Sprott examined five-term systems with only one nonlinear term and found two examples of chaotic systems.

Consider the ordinary differential equation system $\dot{x} = f(x)$, where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is differentiable. We classify the systems as

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1. Uniformly dissipative, i.e. $\nabla \cdot f = \Sigma_{i=1}^{n} \frac{\partial f_i}{\partial x_i} < 0$, for all $x \in \mathbb{R}^n$.

2. Uniformly conservative, i.e. $\nabla \cdot f = 0$, for all $x \in \mathbb{R}^n$.

3. Nonuniformly dissipative and nonuniformly conservative.

In this paper we simply call uniformly conservative and uniformly dissipative systems conservative and dissipative systems respectively.

The authors (Zhang and Heidel, 1997) proved that no three dimensional dissipative quadratic systems of ODE's with a total of four terms on the right-hand side are chaotic. The authors (Heidel and Zhang, 1999) also proved that almost all of the three dimensional conservative quadratic systems of ODE's with four terms on the right-hand side are not chaotic. The only 4-2 system that is not proved rigorously being nonchaotic is $x' = y^2 - z^2,\ y' = x,\ z' = y$. But numerical results indicate that this system does not exhibit chaos. Besides, Yang (2000) and Yang & Chen (2002) resolved analytically over ten nonuniformly conservative and nonuniformly dissipative 4 term quadratic cases. In this paper we consider all conservative three dimensional systems with five terms on the right-hand side and one quadratic term (5-1 systems). This work has already found an application in jerk dynamics, See Linz, S. J. (1997) , Nzotungicimpaye, J. (1994) and Gottlieb, H. P. W. (1998).

Starting from all the 5-1 conservative quadratic systems, after eliminating the linearly equivalent systems, solvable systems, and systems that are equivalent to 2D systems, 4 term systems and/or linear systems, 19 of them left. These 19 systems are listed in section 3 and section 4. The systems that are equivalent to 4 term systems are listed in the appendix.

Each of the systems in this paper is rescaled as in the following example. The system

$$\begin{cases} \dot{x} = ayz + by \\ \dot{y} = cx + rz \qquad\qquad \dot{} = \frac{d}{d\tau} \\ \dot{z} = px \end{cases} \tag{1.1}$$

where a, b, c, r, $p \neq 0$ are constants, can be transformed into one of the following two systems

$$\begin{cases} X' = YZ + AY \\ Y' = \pm X + Z \qquad\qquad ' = \frac{d}{dt} \\ Z' = X \end{cases}$$

by the scalar transformation

$$x = \alpha X, \quad y = \beta Y, \quad z = \gamma Z, \quad \tau = \delta t, \quad \delta > 0$$

for two different sets of parameters a, b, c, r and p. More specifically

$$\alpha = \frac{r^3p^2}{ac^4},\ \beta = \frac{r^2p}{ac^2},\ \gamma = \frac{r^2p^2}{ac^3},\ \delta = \frac{c}{rp}, \quad \text{if } c > 0,\ rp > 0, \text{ or } c < 0,\ rp < 0,$$

and system (1.1) is transformed into the " + " system;

$$\alpha = -\frac{r^3p^2}{ac^4},\ \beta = \frac{r^2p}{ac^2},\ \gamma = \frac{r^2p^2}{ac^3},\ \delta = -\frac{c}{rp}, \quad \text{if } c > 0,\ rp < 0, \text{ or } c < 0,\ rp > 0,$$

and system (1.1) is transformed into the " $-$ " system.

Furthermore the " $+$ " system and the " $-$ " system are related by the following transformation

$$t \mapsto -t,\ X \mapsto X,\ Y \mapsto Y,\ Z \mapsto -Z.$$

We are interested in the asymptotic behavior of the systems for both $\tau \to \infty$ and $\tau \to -\infty$. But it is sufficient to study the solutions for both the " $+$ " and the " $-$ " systems as $t \to \infty$. For convenience (X, Y, Z) is replaced by (x, y, z).

In section 2 we prove a theorem that provides sufficient conditions for 3D systems to be nonchaotic. At the end of Section 2 a conjecture is stated which attempts to pin down the absence of chaos in a more intuitive manner. Section 3 and Section 4 contain the specific proofs for 5-1 conservative systems to be nonchaotic. In section 5 for reader's convenience, we review some of the concepts on chaos and then present some numerical results for the simplest chaotic conservative system.

In all our numerical simulations we use Ermentrout's XPP (Ermentrout) with 4th order Runge-Kutta method and step size $\Delta t = 0.01$.

2 Nonchaotic Behavior

Consider the autonomous system

$$x' = f(x), \quad x \in \mathbb{R}^N, \quad t \in \mathbb{R} \tag{2.1}$$

where $' = \frac{d}{dt}$, $f : \mathbb{R}^N \to \mathbb{R}^N$ is continuous. Let $x(0) = x_0$, and x_j, x_{0j} and f_j, $j = 1, 2, ..., N$ be the jth components of x, x_0 and f respectively. Here we define some terminology that will be used in the proof of the theorem.

We call a solution $x(t)$ of system (2.1) *bounded oscillatory* if it is bounded and there exists an $M_0 > 0$ such that
either for any $\varepsilon > 0$ there exist a t_1 and a t_2 with $|t_1 - t_0| < \varepsilon$ and $|t_2 - t_0| < \varepsilon$ such that $|x_j(t_1) - x_j(t_2)| > M_0$ for some t_0 finite and some $j \in \{1, ..., N\}$,
or for any $T > 0\,(T < 0)$ there exist a $t_1 > T\,(t_1 < T)$ and a $t_2 > T\,(t_2 < T)$ such that $|x_j(t_1) - x_j(t_2)| > M_0$ for some $j \in \{1, ..., N\}$.

Therefore if $x(t)$ is bounded oscillatory, there is a component x_j such that x_j' changes sign infinitely many times and x_j has no limit as $t \to \omega$, $\omega \leqslant \infty$.

We call a solution $x(t)$ *unbounded oscillatory* if for a fixed $M_0 > 0$ and any $M > 0$,
either for any $\varepsilon > 0$ there exist a t_1 and a t_2 with $|t_1 - t_0| < \varepsilon$ and $|t_2 - t_0| < \varepsilon$ such that $|x_j(t_1)| > M$ and $|x_j(t_2)| < M_0$ for some t_0 finite and some $j \in \{1, ..., N\}$
or for any $T > 0(T < 0)$, there exist a $t_1 > T(t_1 < T)$ and a $t_2 > T(t_2 < T)$ such that $|x_j(t_1)| > M$ and $|x_j(t_2)| < M_0$ for some $j \in \{1, ..., N\}$.

A fixed point of system (2.1) is *isolated* if it is also the connected component (Rudin, 1987, pg 197) of the set of fixed points of system (2.1) that contains the fixed point. Then we call the fixed point an $isolated\ fixed\ point$.

Next we recall some concepts in algebra. Let $P(x) = \Sigma_\alpha A_\alpha x^\alpha$ be a polynomial, where $x \in \mathbb{R}^N$, $N \geqslant 1$ is an integer, $\alpha = (\alpha_1, ..., \alpha_N)$, and each of the α_i's is a nonnegative integer, $x^\alpha = x_1^{\alpha_1}...x_N^{\alpha_N}$, the order of the multi-index α is denoted by $|\alpha| = \alpha_1 + ... + \alpha_N$ and $A_\alpha \in \mathbb{R}$. By the fundamental theorem of algebra, every polynomial in the above form can be factored as the product of irreducible polynomials with real coefficients. Therefore for some $m \leqslant |\alpha|$

$$P(x) = \Pi_{j=1}^m P_j(x), \tag{2.2}$$

where $P_j(x)$'s are irreducible polynomials and the zero set $\{(x \in \mathbb{R}^N | P(x) = 0\}$ of $P(x)$ is the union of the zero sets of $P_j(x)$, $j = 1, ..., m$. The zero set of each of the $P_j(x)$ consists of a finite number of connected components and each of them has dimension at most $N - 1$ in $\mathbb{R}^N$. We call each connected component of an irreducible polynomial a $simple\ surface$, a connected component that consists of more than one component of irreducible polynomials a $complex\ surface$. In this paper we also call a connected component an $isolated\ surface$.

For $N = 3$ an isolated surface can consist of a finite number of two dimensional simple surfaces that are joined by a finite number of one dimensional curves and/or points. From now on we consider the case $N = 3$ and make the following assumptions on system (2.1):
$(H1)$ System (2.1) is equivalent to

$$F^+(x_j'', x_j', x_j) = C^+ + \int_0^t G^+(x_j''(s), x_j'(s), x_j(s))\, ds, \quad ' = \frac{d}{dt} \tag{2.3}$$

for some integer $1 \leqslant j \leqslant 3$ and equivalent to

$$F^-(\ddot{x}_j, \dot{x}_j, x_j) = C^- + \int_0^{\tau^*} G^-(\ddot{x}_j(s), \dot{x}_j(s), x_j(s))\, ds, \quad \dot{} = \frac{d}{d\tau^*}, \ \tau^* = -t \tag{2.4}$$

for some integer $1 \leqslant j \leqslant 3$, where $C^\pm$ are constants and $G^\pm(y) \geqslant 0$, for all $y \in \mathbb{R}^3$.
$(H2)$ Each of the functions $f(y)$ in (2.1), $F^\pm(y)$, $G^\pm(y)$ in (2.3) and (2.4), $y \in \mathbb{R}^3$ is either a polynomial or a rational expression such that each of the simple surfaces is homeomorphic to a plane or a sphere or a subset of a plane or a sphere. Let

$$G^\pm(y) = \frac{R_0^\pm(y)}{Q_0^\pm(y)}, \quad F^\pm(y) = \frac{R^\pm(y)}{Q^\pm(y)} \text{ and } f(y) = \frac{r(y)}{q(y)}$$

where $R^\pm(y)$, $Q^\pm(y)$, $r(y)$ and $q(y)$ are polynomials. We assume that $Q_0^\pm(y) > q_0 > 0$, $|Q^\pm(y)| > Q_1 > 0$, and $|q^\pm(y)| > q_1 > 0$ for some positive constants q_0, Q_1, and q_1.

Theorem 2.1 (*Main Theorem*) *Let* $N = 3$. *Under hypotheses* $(H1)$ *and* $(H2)$ *system* (2.1) *has no bounded chaos.*

Usually system (2.1) is equivalent to the following scalar equation of $x_j(t)$:

$$\Phi(x_j^{(N)}, x_j^{(N-1)}, \dots, x_j', x_j) = 0 \tag{2.5}$$

for some $j \in \{1, ..., N\}$, where Φ is a continuous function. Even though there are a large number of systems that satisfy $(H1)$, there is no routine way to transform a system to (2.3) and (2.4). The following well known fact will be applied in the proof of the theorem.

Lemma 2.2 *Let $P_1(x)$ and $P_2(x)$, $x \in \mathbb{R}^3$, be two irreducible polynomials where P_1 and P_2 are not proportional. Then the set $\{x \in \mathbb{R}^3 | P_1(x) = 0$ and $P_2(x) = 0\}$ has dimension zero or one.*

Since we consider only bounded chaos in this paper, for convenience we call bounded chaos simply chaos.

Proof of Theorem 2.1: Chaotic behavior in system (2.1) can happen as $t \to \infty$ and $t \to -\infty$. We show that if (2.3) holds, then the system is not chaotic when $t \to \infty$ and similarly if (2.4) holds, then the system is not chaotic when $t \to -\infty$. Since the proof for the case when $t \to -\infty$ is the same as that when $t \to \infty$, we only prove the case when $t \to \infty$. Without loss of generality we consider solutions for $t \geqslant 0$. For simplicity we denote F^+ and G^+ as F and G respectively. Then $(H1)$ implies that

$$\frac{d}{dt}F(x_j''(t),\, x_j'(t),\, x_j(t)) = G(x_j''(t),\, x_j'(t),\, x_j(t)) \geqslant 0.$$

Therefore either $F(x_j'', x_j', x_j)$ has a limit $L > -\infty$, as $t \to \infty$, where

$$L = C + \int_0^\infty G(x_j''(s),\, x_j'(s),\, x_j(s))\, ds$$

or $F(x_j'', x_j', x_j) \to \infty$, as $t \to \omega < \infty$. For the latter case, $(H2)$ implies that at least one of the x_j, $j = 1, 2, 3$ is unbounded and therefore the solutions are not chaotic. For the first case we will show that the bounded solutions are not chaotic by checking all possible behaviors of the system. Here we classify the solutions by their initial conditions. We first define the following two sets of the initial conditions and consider the solutions in the two sets separately.

$$\Omega_1 = \{(x(0) \in \mathbb{R}^3 \,|\, F(x_j''(t),\, x_j'(t),\, x_j(t)) = \infty, \text{ as } t \to \infty\},$$

$$\Omega_2 = \{(x(0) \in \mathbb{R}^3 \,|\, F(x_j''(t),\, x_j'(t),\, x_j(t)) = L < \infty \text{ as } t \to \infty\}$$

Case 1. $x(0) \in \Omega_1$

Then $(H2)$ implies again that at least one of the x_j, $j = 1, 2, 3$ is unbounded and therefore there can't be chaos in the system when $t \to \infty$.

Case 2. $x(0) \in \Omega_2$

Then $\int_0^\infty G^(x_j''(s),\, x_j'(s),\, x_j(s))\, ds$ is finite. We consider $G \nrightarrow 0$ and $G \to 0$ as $t \to \infty$ in the following two sub cases.

Case 2.1 $G \nrightarrow 0$ as $t \to \infty$.

Since $\int_0^\infty G\, ds$ is bounded, G' is unbounded. $(H2)$ and the fact G' is unbounded imply that at least one of the x_j, $j = 1, 2, 3$ is unbounded. Therefore the solutions are not chaotic. For convenience, we define the set $\Omega_2^\dagger$ as

$$\Omega_2^\dagger = \{x(0) \in \Omega_2 : \lim_{t\to\infty} G = 0\}.$$

Case 2.2 $x(0) \in \Omega_2^\dagger$, i.e. $G \to 0$, as $t \to \infty$.
Then we have:

$$F(x_j''(t), x_j'(t), x_j(t)) \to L \text{ and } G(x_j''(t), x_j'(t), x_j(t)) \to 0 \text{ as } t \to \infty \tag{2.6}$$

Let $\tilde{F}(x) = F(x_j'', x_j', x_j) - L$ and $\tilde{G}(x) = G(x_j'', x_j', x_j)$. Now we define two sets according to the above limits:

$$S_1 := \{x \in \mathbb{R}^3 \,|\, \tilde{F}(x) = 0\,\} \text{ and } S_2 := \{x \in \mathbb{R}^3 \,|\, \tilde{G}(x) = 0\,\}$$

Then by $(H2)$ $\tilde{F}(x)$ and $\tilde{G}(x)$ are polynomials or rational expressions in x. Here we recall the distance between two sets θ_1 and θ_2 in $\mathbb{R}^3$

$$d(\theta_1, \theta_2) = \inf_{x\in\theta_1,\, y\in\theta_2} \| x - y\|$$

where $\|\cdot\|$ is the Euclidian norm. From (2.6) for any solution x with $x(0) \in \Omega_2^\dagger$, $d(x, S_1) \to 0$ and $d(x, S_2) \to 0$, as $t \to \infty$. Since both S_1 and S_2 are close sets, $S_1 \cap S_2$ is not empty and the solutions asymptote to an ω-limit set $\Omega_\omega \subset S_1 \cap S_2$.

Since $\tilde{F}(x)$ and $\tilde{G}(x)$ are polynomials or rational expressions, the connected components of $S_1 \cap S_2$ is a collection of a finite number of points, a finite number of one dimensional curves closed or not closed, a finite number of disjoint two dimensional surfaces and a finite number of unions of one dimensional curves and two dimensional surfaces in $\mathbb{R}^3$. Let n be the total number of connected components of $S_1 \cap S_2$ and denote each of the components I_i, $i = 1, ..., n$. We again classify the initial conditions $x(0) \in \Omega_2^\dagger$ as:

$$\Omega_{2.1}^\dagger = \{x(0) \in \Omega_2^\dagger : \lim_{t\to\infty} d(x(t), I_i) = 0,\ \dim\{I_i\} = 0/1 \text{ for some } i = 1, .., n.\}$$

and

$$\Omega_{2.2}^\dagger = \{x(0) \in \Omega_2^\dagger : \lim_{t\to\infty} d(x(t), I_i) = 0,\ \dim\{I_i\} = 2 \text{ for some } i = 1, .., n.\}.$$

Then $\Omega_{2.1}^\dagger \cup \Omega_{2.2}^\dagger = \Omega_2^\dagger$ and $\Omega_{2.1}^\dagger \cap \Omega_{2.2}^\dagger = \phi$.
Case 2.2.1 $x(0) \in \Omega_{2.1}^\dagger$.
$(H2)$ implies that the one dimensional curves are intersections of the surfaces of irreducible polynomials. Therefore each solution in this case can only either approaches an equilibrium, or a limit cycle or goes to infinity. Therefore the solution can not be chaotic.
Case 2.2.2 $x(0) \in \Omega_{2.2}^\dagger$.

Without loss of generality, we consider when $\tilde{F}(x)$ and $\tilde{G}(x)$ are polynomials. By the fundamental theorem of algebra,

$$\tilde{F}(x) = \Pi_{\tilde{j}=1}^{m_1} \tilde{F}_{\tilde{j}}(x) \text{ and } \tilde{G}(x) = \Pi_{j=1}^{m_2} \tilde{G}_j(x)$$

where $\tilde{F}_{\tilde{j}}(x)$'s and $\tilde{G}_j(x)$'s are irreducible polynomials. Then by $(H2)$ and lemma 2.2 there exists an l such that $\tilde{F}_{\tilde{i}_1}(x) \equiv \tilde{G}_{i_1}(x)$, ..., $\tilde{F}_{\tilde{i}_l}(x) \equiv \tilde{G}_{i_l}(x)$, for all $x \in \mathbb{R}^3$.

We call the set $\{x \in \mathbb{R}^3 | M(x) = 0\}$ positively invariant under the flow $x' = f(x)$ if $M(x(0)) = 0$ implies $M(x(t)) = 0$ for all $t > 0$. Then obviously if $(\nabla M, f)|_{M=0} \equiv 0$, then $M = 0$ is positively invariant under the flow.

Here for each of the I_i with dimension 2, we define

$$d_i = \max_{k=1,..,l_i} \left\{ \dim\{x \in \mathbb{R}^3 | \nabla \tilde{F}_{\tilde{j}_k}(x) \cdot f = 0 \text{ and } \tilde{F}_{\tilde{j}_k}(x) = 0\} \right\}$$

and

$$\Omega^{\dagger}_{2.2.1} = \{x(0) \in \Omega^{\dagger}_{2.2} : \lim_{t\to\infty} d(x(t), I_i) = 0,\ d_i \leqslant 1 \text{ for some } i = 1, .., n.\}$$

and

$$\Omega^{\dagger}_{2.2.2} = \{x(0) \in \Omega^{\dagger}_{2.2} : \lim_{t\to\infty} d(x(t), I_i) = 0,\ d_i = 2 \text{ for some } i = 1, .., n.\}.$$

Case 2.2.2.1 $x(0) \in \Omega^{\dagger}_{2.2.1}$

Clearly each of the solutions in this case will either be asymptotic to a curve which is an intersection of the zero sets of two irreducible polynomials or an equilibrium. Therefore they are not chaotic.

Case 2.2.2.2 $x(0) \in \Omega^{\dagger}_{2.2.2}$

Then each solution approaches an I_i for some $i = 1, ..., n$ that contains a 2D invariant set which can be topologically equivalent to a torus or a more complicated surface on which there could be chaotic behavior. By (H2) such surfaces must be the union of the simple surfaces. However if the zero sets of two irreducible polynomials are both invariant, their intersection is also invariant. By lemma 2.2, the intersection has dimension at most one. This means that solutions can't switch from the zero set of one irreducible polynomial to another. By the Poincar*é*-Bendixon Theorem on 2-manifolds (Hartman, 1964), the solutions on the 2D surface can only be or approach an equilibrium, a periodic orbit or it is unbounded. Therefore they are not chaotic. Each of the solutions that approach the 2D surface will either stay on or approach only one of the zero set of an irreducible polynomial. By (H2) the solutions are not chaotic. This completes the proof of the theorem. ■

Note that by our theorem no chaotic 3D systems with polynomial right-hand sides satisfy both $(H1)$ and $(H2)$.

One of the goals of this paper is to attempt to develop a general method to determine if a nonlinear autonomous system is nonchaotic. It is well known that it is difficult to give a rigorous general definition for chaos in a mathematical sense for the solutions of dynamical systems because of the complexity of the topological structures of the chaotic solutions. Brown and Chua (Brown and Chua, 1996) listed nine definitions of chaos. As far as we know thirteen definitions have been given so far, but none of them can be the final version of a general definition. This means that nonchaotic behavior of solutions hasn't yet been defined in a satisfactory manner either. Nevertheless we can give the following criterion as a conjecture for recognizing nonchaotic behavior:

Conjecture 2.3 *Criterion: An N dimensional system* (2.1) *with no cluster points in the set of isolated fixed points has no bounded chaos if for any of its solutions there are $N-2$ components $x_{n_k}(t)$, $n_k \in \{1, ..., N\}$ and $k = 1, ..., N-2$, such that for each of the $N-2$ components only the following cases can happen:*
as $t \to \infty$, similarly as $t \to -\infty$,

- (i) *It tends to a finite limit,*
- (ii) *It is periodic or asymptotic to a periodic function,*
- (iii) *It is unbounded;*

there exists an ω, $|\omega| < \infty$ such that,

- (iv) *It is unbounded, as $t \to \omega$,*
- (v) *It is bounded but does not have a limit, as $t \to \omega$,*
- (vi) *It is bounded and has a limit as $t \to \omega$ but not defined at $t = \omega$.*

Note that the difference between (iii) and (iv) is that the solutions in (iii) are defined for all $t_0 \leqslant t < \infty$ or $-\infty < t \leqslant t_0$ while the solutions in (iv) are not defined at $t = \omega < \infty$, $|\omega| < \infty$. For example (a) if $x_j(t) = \sin(\frac{1}{t})$ for some $j \in \{1, ..., N\}$, then $x_j(t)$ does not have a limit at $t = 0$ and it is bounded as $t \to 0$ and it has a limit as $t \to \infty$; (b) if $x_j(t) = e^{\frac{1}{t}} \sin(\frac{1}{t})$ for some $j \in \{1, ..., N\}$, then $x_j(t)$ does not have a limit at $t = 0$ and it is unbounded as $t \to 0$, it is bounded oscillatory (see definition below) as $t \to \infty$. (c) chaotic solutions of the Lorenz equations, which are proved to exist analytically by Hastings and Troy (1996), do not satisfy our criterion 2.3. (d) If the Duffing equation is considered as a 3D autonomous system, the component t of the solution must go to infinity. It is proved by Ai and Hastings (2002) that Duffing equation with certain forcing has chaotic solutions.

For convenience, we don't make a distinction between the notations $\dot{x} = x' = \frac{dx}{dt}$ in the next 2 sections.

3 Five-term conservative systems with one quadratic term and without constant terms.

All the systems without constant terms that need to be considered are the following 11 of the 19 systems mentioned in section 1.

$$\begin{cases} x' = y^2 + Ay + z \\ y' = x \\ z' = \pm y \end{cases} \tag{3.1}$$

$$\begin{cases} x' = yz \pm y + Az \\ y' = x \\ z' = y \end{cases} \tag{3.2}$$

$$\begin{cases} x' = y^2 + Az \\ y' = x \pm z \\ z' = x \end{cases} \tag{3.3}$$

$$\begin{cases} x' = y^2 + Az \\ y' = x \pm z \\ z' = y \end{cases} \tag{3.4}$$

$$\begin{cases} x' = z^2 + Ay \\ y' = x \pm z \\ z' = x \end{cases} \tag{3.5}$$

$$\begin{cases} x' = z^2 + Ay \\ y' = x \pm z \\ z' = y \end{cases} \tag{3.6}$$

$$\begin{cases} x' = z^2 + Az \\ y' = x \pm z \\ z' = y \end{cases} \tag{3.7}$$

$$\begin{cases} x' = yz + Ay \\ y' = \pm x + z \\ z' = x \end{cases} \tag{3.8}$$

$$\begin{cases} x' = yz + Az \\ y' = x + z \\ z' = \pm y \end{cases} \tag{3.9}$$

$$\begin{cases} x' = y + z \\ y' = \pm x + Az \\ z' = x^2 \end{cases} \tag{3.10}$$

$$\begin{cases} x' = y + z \\ y' = \pm x + Az \\ z' = xy \end{cases} \tag{3.11}$$

where the " + " and " − " correspond to (3.xa) and (3.xb) respectively and x represents one of the positive integers 1, 2,..., 11.

Theorem 3.1 *Systems* (3.1)-(3.7a), (3.9), (3.10) *and* (3.11a) *are not chaotic.*

Proof: Systems (3.1)-(3.7a), (3.9)-(3.10) can be written as the following scalar equations.

$$(3.1)\pm \quad \dddot{z} = \pm \dot{z}^2 + A\dot{z} \pm z, \text{ multiplied by } \ddot{z} \text{ and integrating}$$

$$\ddot{z}^2 \mp \frac{2}{3}\dot{z}^3 \mp A\dot{z}^2 \mp 2z\dot{z} = C \mp 2\int_0^t \dot{z}^2(s)\,ds$$

$$(3.2)\pm \quad \dddot{z} = \dot{z}z \pm \dot{z} + Az, \text{ multiplied by } z \text{ and integrating}$$

$$\ddot{z}z - \frac{1}{2}\dot{z}^2 - \frac{1}{3}z^3 \mp \frac{1}{2}z^2 = C + \int_0^t Az^2(s)\,ds$$

$$(3.3)\pm \quad \dddot{y} = 2y\dot{y} + A\dot{y} \pm y^2 \quad \text{ integrating}$$

$$\ddot{y} - y^2 - Ay = C \pm \int_0^t y^2(s)\,ds$$

$$(3.4)\pm \quad \dddot{y} = 2y\dot{y} \pm \dot{y} + Ay, \text{ multiplied by } y \text{ and integrating}$$

$$\ddot{y}y - \frac{1}{2}\dot{y}^2 - \frac{2}{3}y^3 \mp \frac{1}{2}y^2 = C + \int_0^t Ay^2(s)\,ds$$

$$(3.5)\pm \quad \dddot{z} = 2z\dot{z} + A\dot{z} \pm Az, \text{ multiplied by } z \text{ and integrating}$$

$$\ddot{z}z - \frac{1}{2}\dot{z}^2 - \frac{A}{2}z^2 - \frac{2}{3}z^3 = C \pm \int_0^t Az^2(s)\,ds$$

$$(3.6)\pm \quad \dddot{z} = z^2 + (A \pm 1)\dot{z} \quad \text{integrating}$$

$$\ddot{z} - (A \pm 1)z = C + \int_0^t Az^2(s)\,ds$$

$$(3.7a) \quad \dddot{z} = z^2 + \dot{z} + Az, \text{ multiplied by } \dot{z} \text{ and integrating}$$

$$\ddot{z}\dot{z} - \frac{A}{2}z^2 - \frac{1}{3}z^3 = C + \int_0^t (\ddot{z}^2(s) + \dot{z}^2(s))\,ds$$

$$(3.9)\pm \quad \dddot{z} = \dot{z}z \pm \dot{z} \pm Az, \text{ multiplied by } z \text{ and integrating}$$

$$\ddot{z}z - \frac{1}{2}\dot{z}^2 - \frac{1}{3}z^3 \mp \frac{1}{2}z^2 = C \pm \int_0^t Az^2(s)\,ds$$

$$(3.10)\pm \quad \dddot{x} = 2x\dot{x} \pm \dot{x} + Ax^2 \quad \text{integrating}$$

$$\ddot{x} - x^2 \mp x = C + \int_0^t Ax^2(s)\,ds$$

where $A \neq 0$ and C are arbitrary constants. Then systems (3.1) – (3.7a), (3.9), and (3.10) satisfy hypotheses ($H1$) and ($H2$). By theorem 2.1 none of them is chaotic.

Now we look at system (3.11a). If $A = 1$ we have that $x' - y' = -(x - y)$, and $x(t) = y(t) + Ce^{-t}$. Then

$$x'' - x' = Ce^{-t} + x^2 - Cxe^{-t} \tag{3.12}$$

Let $X(t) = x(t) - \frac{C}{2}e^{-t}$. Equation (3.12) becomes

$$X'' - X' - X^2 = -\frac{1}{4}C^2e^{-2t}$$

$$X'' - X' - X^2 \to x'' - x' - x^2 = 0, \qquad \text{as } t \to \infty$$

Therefore hypotheses ($H1$) and ($H2$) are satisfied and system (3.11a) is not chaotic for $A = 1$. If $A = -1$, let $u = x + y = Ce^t$, then $\frac{1}{2}(x+y)^2 = \int_0^t (x+y)^2 + C$. Therefore hypotheses ($H1$) and ($H2$) are satisfied and by theorem 2.1 system (3.11a) is not chaotic either for $A = -1$. For $A \neq \pm 1$, from $y' = x + A(x' - y)$, we have

$$(y - Ax)' = x - Ay = -A(y - Ax) + (1 - A^2)x$$

Let $u = y - Ax$, $a = 1 - A^2$. Then $u' + Au = ax$. Since $A \neq \pm 1$, $z' = xy = x'' - y'$, multiplied by a to get

$$(u' + Au)y = u''' + Au'' - ay'$$

$$y = u + Ax = u + \frac{A}{a}(u' + Au)$$

then we have

$$(u' + Au)\left(u + \frac{A}{a}(u' + Au)\right) = u''' + Au'' - a\left(u' + \frac{A}{a}(u'' + Au')\right)$$

$$uu' + \frac{A}{a}\left((1 - A^2)u^2 + (u')^2 + 2Auu' + A^2u^2\right) = u''' - u',$$

and thus

$$u''' - u' - \left(1 + \frac{2A^2}{a}\right)uu' = \frac{A}{a}\left(u^2 + (u')^2\right)$$

Integrate the above equation to get

$$u'' - u - \frac{1}{2}\left(1 + \frac{2A^2}{a}\right)u^2 = C + \frac{A}{a}\int_0^t \left(u^2(s) + u'^2(s)\right)\,ds \tag{3.13}$$

where C is an arbitrary constant. Equation (3.13) satisfies hypotheses $(H1)$ and $(H2)$ and by theorem 2.1 the system is not chaotic when $A \neq \pm 1$. Hence system $(3.11a)$ is not chaotic. This completes the proof of Theorem 3.1. $\Box$

For $(3.7b)$, we have the following proposition:

Proposition 3.2 *The scalar equation in* $z + \frac{A}{2}$ *of system* $(3.7b)$ *with* $A \in \mathbb{R}$ *is linearly equivalent to the scalar equation in* y *of system* $(4.1b)$ *with* $A < 0$.

Proof : A scalar equation of $(3.7b)$ is given by $z''' = z^2 - z' + Az$. Let $Z = z + \frac{A}{2}$, $Y = y$ and $X = x$. Then $(3.7b)$ becomes $X' = Z^2 - (\frac{A}{2})^2$, $Y' = X - Z + \frac{A}{2}$, and $Z' = Y$ and so $Z''' = X' - Z' = Z^2 - (\frac{A}{2})^2 - Z'$. While the scalar equation of $(4.1b)$ in y is $y''' = y^2 - y' + A$. Hence it's equivalent to $(4.1b)$ for $A < 0$. $\Box$

Consider system $(3.8a)$

$$\begin{cases} x' = yz + Ay, & A \neq 0,\, t \geqslant 0 \\ y' = x + z, & z' = x; \end{cases}$$

then we have the following theorem:

Theorem 3.3 *System* $(3.8a)$ *has no chaos for* $A > 0$.

We first prove the following lemmas.

Lemma 3.4 *If* $A < 0\,(A > 0)$*, then any bounded solutions to* $(3.8a)$ *with* $z(t)$ *oscillatory satisfy*

$$\int_0^\infty y(t)\,dt = \int_0^\infty u(t)\,dt = \infty(-\infty)$$

where $u = y - z$.

Proof: Let $u = y - z$. Then $u' = z$ and the scalar equation in u can be written as:

$$u''' = uu' + (u')^2 + Au' + Au. \tag{3.14}$$

Integrate (3.14) to get

$$\int_0^t u(s)\,ds = \frac{1}{A}(C + u'' - \frac{1}{2}u^2 - Au) - \frac{1}{A}\int_0^t (u'(s))^2\,ds.$$

z is bounded oscillatory and $u' = z$ imply that u' and u'' are also bounded oscillatory. Then the right-hand side of the above equation goes to $\infty(-\infty)$ for $A < 0(A > 0)$ and so does the left-hand side. Integrate $u' = z$ to get $u = u(0) + \int_0^t z(s)\,ds$. Obviously $\int_0^t z(s)\,ds$ is bounded for all $t \geqslant 0$. Again using $y - z = u$, we have

$$\int_0^\infty y(s)\,ds = \int_0^\infty z(s)\,ds + \int_0^\infty u(s)\,ds = \infty(-\infty)$$

for $A < 0\,(A > 0)$. This completes the proof of this lemma. $\Box$

Lemma 3.5 *In system (3.8a), there exist no bounded solution with $z(t)$ oscillatory that have the property that there exists a $t_1 > 0$ such that $z(t)$ doesn't change sign for $t \geqslant t_1$.*

Proof: From (3.8a) we have

$$(z+A)\dddot{z} - \dot{z}\ddot{z} = (z+A)^2(\dot{z}+z). \tag{3.15}$$

Then integrate (3.15) to get

$$z\ddot{z} - \dot{z}^2 + A\ddot{z} - \frac{1}{3}(z+A)^3 = C + \int_0^t z(s)(z(s)+A)^2\,ds. \tag{3.16}$$

$z(t)$ being bounded oscillatory implies that the right-hand side of (3.16) goes to $\infty(-\infty)$ but its left-hand side is bounded, a contradiction. This completes the proof of this lemma. $\square$

Note that from the above lemma if $z \in C^1(\mathbb{R})$ is bounded oscillatory, then $z(t)$ crosses zero infinitely many times.

Proof of Theorem 3.3. Suppose $p(t)$ is a bounded solution of (3.8a). If $z(t)$ has a limit z^* as $t \to \infty$, then the solution approaches the plane $z = z^*$ and therefore the invariant set on $z = z^*$. By the Poincaré-Bendixon Theorem the solution is not chaotic. If $z(t)$ is oscillatory then by Lemma 3.4 $\int_0^\infty y(t)\,dt = -\infty$ for $A > 0$. From $\ddot{z} = y(z+A)$

$$\int_0^t \frac{\ddot{z}(s)}{z(s)+A}\,ds = \int_0^t y(s)\,ds.$$

An integration by parts gives that

$$\frac{\dot{z}(t)}{z(t)+A} + \int_0^t \left(\frac{\dot{z}(s)}{z(s)+A}\right)^2 ds = \frac{\dot{z}(0)}{z(0)+A} + \int_0^t y(s)\,ds. \tag{3.17}$$

We consider the following cases.

Case 1. $z(0) + A \neq 0$.

Case 1.1. There exists a $T > 0$ such that for all $t > T$ $|z(t)+A| > A_0$ for some $A_0 > 0$. Then $\frac{\dot{z}(t)}{z(t)+A}$ is bounded for $t > T$. But

$$-\int_0^\infty \left(\frac{\dot{z}(s)}{z(s)+A}\right)^2 ds + \frac{\dot{z}(0)}{z(0)+A} + \int_0^\infty y(s)\,ds = -\infty$$

a contradiction to (3.17).

Case 1.2. $z(t)$ crosses $-A$ infinitely many times

Then there exists an increasing sequence $\{t_k\}_{k=1}^\infty$ with $t_k \to \infty$ as $k \to \infty$ such that $z(t_k)+A = 0$, for $k = 1, 2, \ldots$. Together with lemma 3.5 we have that for any $M > \left|\frac{\dot{z}(0)}{z(0)+A}\right|$ there exists a $t^* \in (t_k, t_{k+1})$ for some k such that

$$\dot{z}(t^*) = 0,\ z(t^*) + A \neq 0, \text{ and } \int_0^{t^*} y(t)\,dy < -M$$

But (3.17) gives

$$\frac{\dot{z}(0)}{z(0)+A} = \int_0^{t^*} \left(\frac{\dot{z}(s)}{z(s)+A}\right)^2 ds - \int_0^{t^*} y(s)\,ds > \int_0^{t^*} \left(\frac{\dot{z}(s)}{z(s)+A}\right)^2 ds + M,$$

a contradiction.

Case 1.3. There exists an increasing sequence $\{t_k\}$ such that $z(t_k) + A \to 0$ as $t_k \to \infty$.

Then $\forall \varepsilon > 0$, $\exists\ N > 0$, such that $k > N$ implies that $|z(t_k) + A| < \varepsilon$. By lemma 3.5 $z(t)$ must cross zero infinitely many times. Therefore there exists a $t^* > t_N$ such that $\dot{z}(t^*) = 0$, $z(t^*) > 0$ and $z(t^*) + A > 0$. Using the same argument as in case 1.2, it follows that this is impossible.

Case 2. $z(0) + A = 0$.

By lemma 3.5 there exists a $t_1 > 0$ such that $z(t_1) = 0$. Then the proof in case 1 can be applied. Therefore $z(t)$ can't be bounded oscillatory either in this case.

Then either $z(t)$ is unbounded oscillatory or $z(t)$ is bounded but at least one of x and y is unbounded. This completes the proof of the theorem. $\square$

Consider system (3.8b)

$$\begin{cases} x' = yz + Ay, \quad A \neq 0,\, t \geqslant 0 \\ y' = -x + z, \quad z' = x. \end{cases}$$

We have the following theorem:

Theorem 3.6 *System* (3.8b) *is not chaotic for* $A < 0$.

Lemma 3.7 *If* $A < 0\,(A > 0)$*, then any bounded solutions to* (3.8a) *with* $z(t)$ *oscillatory have that:*

$$\int_0^\infty y(t)\,dt = \int_0^\infty v(t)\,dt = -\infty(\infty)$$

where $v = y + z$.

Proof: Let $v = y + z$. Then $v' = z$. The scalar equation in v can be obtained as:

$$v''' = vv' - (v')^2 - Av' + Av \tag{3.18}$$

Integrating (3.18) we have

$$\int_0^t v(s)\,ds = \frac{1}{A}(C + v'' - \frac{1}{2}v^2 + Av) + \frac{1}{A}\int_0^t (v'(s))^2\,ds.$$

If $z(t)$ is bounded oscillatory then so are v, v' and v'' because $v' = z$. Therefore $\frac{1}{A}\int_0^\infty (v'(s))^2\,ds = \infty(-\infty)$ for $A > 0\,(A < 0)$. By the above equation $\int_0^\infty v(s)\,ds = \infty(-\infty)$. Integrate $v' = z$ to get $v = v(0) + \int_0^t z(s)\,ds$. Obviously $\int_0^t z(s)\,ds$ is bounded for all $t \geqslant 0$. Again using $y + z = v$ we have

$$\int_0^\infty y(s)\,ds = -\int_0^\infty z(s)\,ds + \int_0^\infty v(s)\,ds = \infty(-\infty)$$

when $A > 0\,(A < 0)$. Hence $\int_0^\infty y(s)\,ds = \infty(-\infty)$. This completes the proof of the lemma. $\square$

Lemma 3.8 *In system (3.8b), there exist no bounded solution with $z(t)$ oscillatory that have the property that there exists a $t_1 > 0$ such that $z(t)$ doesn't change sign for $t \geqslant t_1$.*

Proof: The result follows from

$$z\ddot{z} - \dot{z}^2 + A\ddot{z} + \frac{1}{3}(z+A)^3 = C + \int_0^t z(s)(z(s)+A)^2\, ds. \tag{3.19}$$

□

Proof of Theorem 3.6: Since $\ddot{z} = y(z+A)$ an integration by part leads to

$$\frac{\dot{z}(t)}{z(t)+A} + \int_0^t \left(\frac{\dot{z}(s)}{z(s)+A}\right)^2 ds = \frac{\dot{z}(0)}{z(0)+A} + \int_0^t y(s)\, ds. \tag{3.20}$$

Suppose that $z(t)$ is bounded oscillatory. Then $\int_0^\infty y(s)\, ds = -\infty$ when $A < 0$. Applying the same argument as in the proof of theorem 3.3 completes the proof of this theorem. □

For the case $A > 0$ in (3.8b), let $p_0 = (x(0), y(0), z(0)) \in \mathbb{R}^3$, we have the following theorem.

Theorem 3.9 *If $A \geqslant \frac{3}{16}$ and $p_0 \in S_3$, where*

$$S_3 = \{(x,\, y,\, z) |\, x - (y-z)z + \frac{1}{3}(y-z)^3 + \frac{1}{2}(y-z)^2 + A(y-z) - A \geqslant 0\}$$

then there exist no bounded oscillatory solutions in system (3.8b) for $t \geqslant 0$.

Proof. Let $x = -X$, $y = -Y$, $z = -Z$. Then (3.8b) becomes

$$\begin{cases} X' = -YZ + AY \qquad A \neq 0,\, t \geqslant 0 \\ Y' = -X + Z, \quad Z' = X. \end{cases} \tag{3.21}$$

Let $u = Y + Z$, $u' = Z$. Then we have the scalar equation in u:

$$u''' = -uu' + (u')^2 - Au' + Au \tag{3.22}$$

Multiply (3.22) by $e^{\int_0^t u(s)\, ds}$. Since

$$\begin{aligned} u''' e^{\int_0^t u(s)\, ds} &= \left(u'' e^{\int_0^t u(s)\, ds}\right)' - \left(uu' e^{\int_0^t u(s)\, ds}\right)' + \left(\frac{1}{3}u^3 e^{\int_0^t u(s)\, ds}\right)' \\ &\quad + (u')^2 e^{\int_0^t u(s)\, ds} - \frac{1}{3}u^4 e^{\int_0^t u(s)\, ds} \\ u'u e^{\int_0^t u(s)\, ds} &= \left(\frac{1}{2}u^2 e^{\int_0^t u(s)\, ds}\right)' - \frac{1}{2}u^3 e^{\int_0^t u(s)\, ds} \\ u' e^{\int_0^t u(s)\, ds} &= \left(u e^{\int_0^t u(s)\, ds}\right)' - u^2 e^{\int_0^t u(s)\, ds} \\ u e^{\int_0^t u(s)\, ds} &= \left(e^{\int_0^t u(s)\, ds}\right)' \end{aligned}$$

we have

$$u'' - uu' + \frac{1}{3}u^3 + \frac{1}{2}u^2 + Au - A = C^* e^{-\int_0^t u(s)\,ds} + \frac{1}{3}e^{-\int_0^t u(s)\,ds}\int_0^t u^2(s)\left(u^2(s) + \frac{3}{2}u(s) + 3A\right)e^{\int_0^s u(\tau)\,d\tau}\,ds \tag{3.23}$$

where $C^* = u''(0) - u(0)u'(0) + \frac{1}{3}u^3(0) + \frac{1}{2}u^2(0) + Au(0) - A$. We can see that if $A \geqslant \frac{3}{16}$ then $u^2(t) + \frac{3}{2}u(t) + 3A \geqslant 0$ for all $t \geqslant 0$. Let

$\Phi(t) = u'' - uu' + \frac{1}{3}u^3 + \frac{1}{2}u^2 + Au - A$ and

$\varphi(t) = \frac{1}{3}e^{-\int_0^t u(s)\,ds}\int_0^t u^2(s)\left(u^2(s) + \frac{3}{2}u(s) + 3A\right)e^{\int_0^s u(\tau)\,d\tau}\,ds$

Then

$$\Phi(t) = C^* e^{-\int_0^t u(s)\,ds} + \varphi(t).$$

Since $u = Y + Z = -y - z$ and $A \geqslant \frac{3}{16}$, by lemma 3.7 $\int_0^\infty u(s)\,ds = -\infty$. Clearly if $C^* > 0$, $C^* e^{-\int_0^\infty u(s)\,ds} = \infty$. When $A \geqslant \frac{3}{16}$ $\varphi(t) \geqslant 0$ for $t \geqslant 0$, and so the right-hand side of (3.23) goes to ∞. But the boundedness of u, u', u'' implies that $\Phi(t)$ is bounded for $t \geqslant 0$, a contradiction. If $C^* = 0$, then clearly $\varphi(\infty) = \infty$. But $\Phi(t)$ is bounded for $t \geqslant 0$, a contradiction. This completes the proof of the theorem. □

For the case when $A < 0$ in system (3.8a), and the cases when $0 < A < \frac{3}{16}$, $p_0 \in \mathbb{R}^3$ and $A > \frac{3}{16}$, $p_0 \in \mathbb{R}^3 \setminus S_3$ in system (3.8b), numerical simulations show that solutions can't be more complicated than the ones shown in figures 1 and 2. The solutions can be oscillatory at the beginning. They will eventually approach a limit or a stable periodic orbit or become unbounded. This suggests that there is no chaos in (3.8).

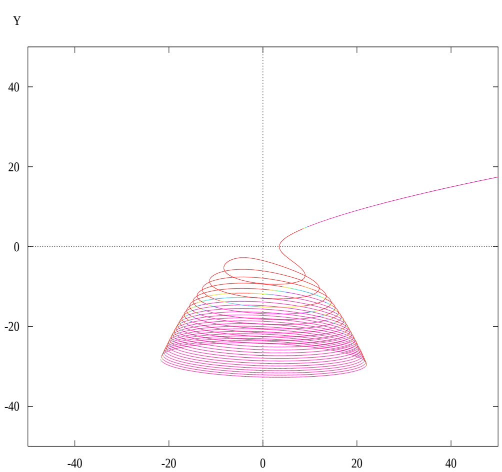

Figure 1: (3.8a) phase space, $x(0) = -21.746$ $y(0) = -28.247$, $z(0) = -0.0125$, $A = -0.15$

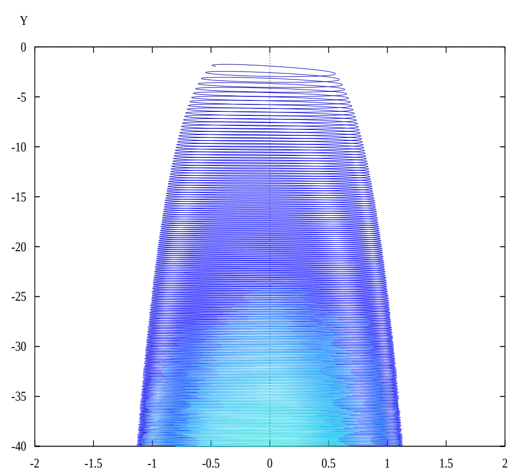

Figure 2: (3.8b) phase space, $x(0) = -0.46$ $y(0) = -1.95$, $z(0) = -0.012$, $A = 0.14$

Numerical simulations on system (3.11b), see figure 3 to figure 6, indicate that when $A > 0$ but small the solutions approach an invariant manifold which looks like a wool-thread hat;

when $A < 0$, solutions oscillate but grow in magnitude faster. When $|A|$ is large, the behaviors are simpler than those when $|A|$ is small. The solutions in figures 3 and 4 have the same initial conditions and the same value of A. Figure 3 is the 'tip' part of figure 4. It appears that all solutions are oscillatory but become either unbounded or approach a periodic solution eventually. No chaotic behaviors have been found in this system.

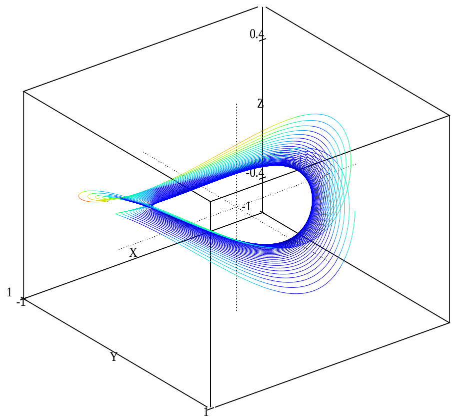

Figure 3: Equation (3.11b), $x(0) = -0.6$, $y(0) = 0.5$, $z(0) = -0.0125$, $A = 0.125$ t=0..300

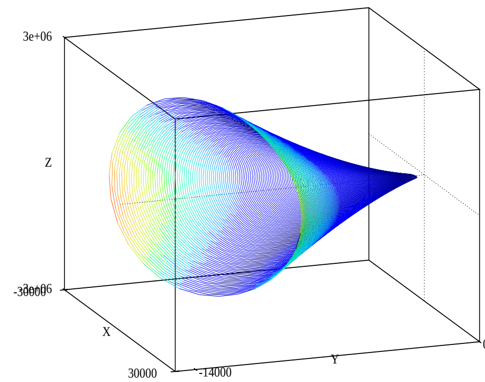

Figure 4: Equation (3.11b), $x(0) = -0.6$, $y(0) = 0.5$, $z(0) = -0.0125$, $A = 0.125$ t=0..600

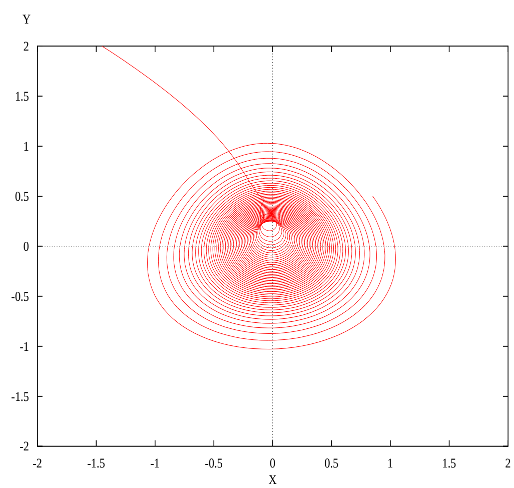

Figure 5: Equation (3.11b), $x(0) = 0.85$, $y(0) = 0.5$, $z(0) = 0.0125$, $A = 0.125$

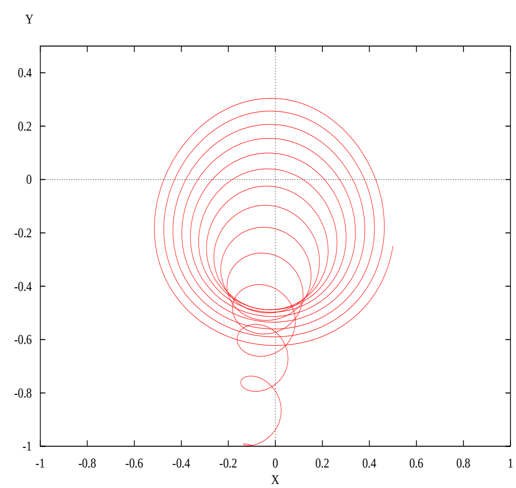

Figure 6: Equation (3.11b), $x(0) = 0.5$, $y(0) = -0.25$, $z(0) = 0.16$, $A = -0.125$

Next we consider the following 5-1 systems which can be conservative and dissipative for different choices of the parameter A.

$$\begin{cases} x' = y^2 + Ax + z \\ y' = x \\ z' = \pm z \end{cases} \tag{3.24}$$

$$\begin{cases} x' = Ax + y + z \\ y' = x^2 \\ z' = \pm z \end{cases} \tag{3.25}$$

$$\begin{cases} x' = y^2 + Ax \\ y' = x + z \\ z' = \pm z \end{cases} \tag{3.26}$$

$$\begin{cases} x' = y^2 \pm x \\ y' = Ay + z \\ z' = x \end{cases} \tag{3.27}$$

$$\begin{cases} x' = y^2 + z \\ y' = x + Ay \\ z' = \pm z \end{cases} \tag{3.28}$$

$$\begin{cases} x' = z^2 + Ax \\ y' = x \pm y \\ z' = y \end{cases} \tag{3.29}$$

$$\begin{cases} x' = yz + Ax \\ y' = x \pm y \\ z' = x \end{cases} \tag{3.30}$$

$$\begin{cases} x' = yz + Ax \\ y' = \pm y + z \\ z' = x \end{cases} \tag{3.31}$$

$$\begin{cases} x' = \pm x + y \\ y' = Ay + z \\ z' = x^2 \end{cases} \tag{3.32}$$

$$\begin{cases} x' = \pm x + y \\ y' = Ay + z \\ z' = xy \end{cases} \tag{3.33}$$

$$\begin{cases} x' = \pm x + z \\ y' = Ay + z \\ z' = xy \end{cases} \tag{3.34}$$

Remark 3.10 *Systems* (3.24), (3.25), (3.26), (3.27), (3.28), (3.29), (3.30), (3.31), (3.32), (3.33) *and* (3.34) *are conservative when* $A = \mp 1$.

Here we have the following theorem for any parameter A (Zhang and Heidel, 2006).

Theorem 3.11 *Systems* (3.24) $\backsim$ (3.33) *are not chaotic.*

Proof : Since

$$\begin{aligned} &(3.24) \quad \ddot{y} - y^2 - A\dot{y} = Ce^{\pm t} \\ &(3.25) \quad \ddot{x} - x^2 - Ax = \pm Ce^{\pm t} \\ &(3.26) \quad \ddot{y} - y^2 - A\dot{y} = (\pm A - A^2)Ce^{\pm t} \\ &(3.28) \quad \ddot{y} - y^2 = (A+1)Ce^{\pm t} \end{aligned}$$

where $A \neq 0$, and C is an arbitary constant, each of the above systems is asymptotic to a 2D surface which is topologically equivalent to a plane or a subset of a plane or unbounded. Therefore there is no bounded chaos in these systems. Now we look at the remaining systems.

$$\begin{aligned} (3.27)\pm: &\quad \dddot{y} = y^2 + (A \pm 1)\ddot{y} \mp A\dot{y}, \quad \text{or} \\ \pm: &\quad \ddot{y} - (A \pm 1)\dot{y} \pm y = C + \int_0^t y^2(s)\,ds \\ (3.29)\pm: &\quad \dddot{z} = z^2 + (A \pm 1)\ddot{z} \mp A\dot{z}, \quad \text{or} \\ \pm: &\quad \ddot{z} - (A \pm 1)\dot{z} \pm Az = C + \int_0^t z^2(s)\,ds \end{aligned}$$

In system (3.30), let $u = y - z$, $\dot{u} = \pm y$. Then its scalar equation is

$$\dddot{u} \mp (A+1)\ddot{u} + u\dot{u} \pm A\dot{u} = \pm \dot{u}^2,$$

integrate to get

$$\ddot{u} \mp (A+1)\dot{u} + \frac{1}{2}u^2 \pm Au = C \pm \int_0^t \dot{u}^2(s)\,ds$$

$$\begin{aligned} (3.31)\pm: &\quad \dddot{y} = \mp y^2 \pm \ddot{y} + y\dot{y} + A(\dot{y} \mp \dot{y}) \quad \text{or} \\ \pm: &\quad \ddot{y} - \frac{1}{2}y^2 - A(y \pm y) = C \mp \int_0^t y^2(s)\,ds \\ (3.32)\pm: &\quad \dddot{x} = x^2 + (A \pm 1)\ddot{x} \mp A\dot{x}, \quad \text{or} \\ \pm: &\quad \ddot{x} - (A \pm 1)\dot{x} \pm Ax = C + A\int_0^t x^2(s)\,ds \end{aligned}$$

$$(3.33)\pm: \quad \dddot{x} = \mp x^2 + (A \pm 1)\ddot{x} \mp A\dot{x} + x\dot{x}, \qquad \text{or}$$

$$\pm: \quad \ddot{x} - (A \pm 1)\dot{x} \pm Ax - \frac{1}{2}x^2 = C \mp \int_0^t x^2(s)\, ds$$

By theorem 2.1, systems (3.27), (3.29), (3.30), (3.31), (3.32), and (3.33) are not chaotic either. □

Consider system (3.34) for $A = -1$. Then the system

$$\begin{cases} x' = x + z \\ y' = -y + z \\ z' = xy \end{cases}$$

is conservative. The system has one equilibrium point (0, 0, 0) and two 1D invariant manifolds $x(t) = 0$, $y(t) = C_1 e^{-t}$, $z(t) = 0$ and $x(t) = C_2 e^t$, $y(t) = 0$, $z(t) = 0$.

Note that the system when $A = -1$ is linearly equivalent to the system when $A = 1$ by exchanging x and y. Differentiate the last equation to get $z'' = z(x - y)'$ and so $\frac{z''}{z} = (x - y)'$. Integrate both sides to get

$$\frac{z'}{z} - x + y = C - \int_0^t \left(\frac{z'}{z}\right)^2 ds \tag{3.35}$$

Then $(\frac{z'}{z} - x + y)' \leqslant 0$. If z crosses zero a finite number of times, then obviously the system is not chaotic by theorem 2.1. Numerical simulations show that there are solutions in which $z(t)$ crosses zero infinitely many times but then z and y approach zero as t goes to infinity. See figure 7.

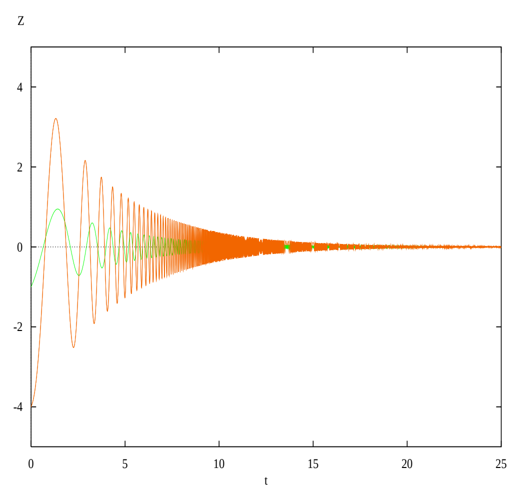

Figure 7: Equation (3.34), $x(0) = -1$, $y(0) = -1$, $z(0) = -1$ and $z(0) = -4$

4 Five-term conservative systems with one quadratic term and constant terms.

All three-dimensional 5-1 conservative systems with one quadratic term and constant terms that needs to be considered are the 8 of the 19 systems mentioned in section 1. They are listed in the following:

$$\begin{cases} x' = y^2 \pm z + A \\ y' = z \\ z' = x \end{cases} \tag{4.1}$$

$$\begin{cases} x' = y^2 \pm z \\ y' = x + A \\ z' = x \end{cases} \tag{4.2}$$

$$\begin{cases} x' = z^2 \pm y \\ y' = x + A \\ z' = x \end{cases} \tag{4.3}$$

$$\begin{cases} x' = y^2 + A \\ y' = x + z \\ z' = \pm x \end{cases} \tag{4.4}$$

$$\begin{cases} x' = y^2 \pm z \\ y' = x + z \\ z' = A \end{cases} \tag{4.5}$$

$$\begin{cases} x' = z^2 + A \\ y' = x \pm z \\ z' = y \end{cases} \tag{4.6}$$

$$\begin{cases} x' = yz + A \\ y' = x \pm z \\ z' = x \end{cases} \tag{4.7}$$

$$\begin{cases} x' = yz + A \\ y' = x \pm z \\ z' = y \end{cases} \tag{4.8}$$

where " + " and " − " correspond to $(4.na)$ and $(4.nb)$ respectively and n represents one of positive integers 1, 2, ..., 8.

Theorem 4.1 *Systems* (4.1a), (4.2) $\sim$ (4.6a), (4.8) *are not chaotic.*

Proof. The scalar equation of system (4.1a) in y is $\dddot{y} = y^2 + \dot{y} + A$. Multiply by $\dot{y}$ and integrate to get

$$\dot{y}\ddot{y} - \frac{1}{3}y^3 - Ay = C + \int_0^t (\ddot{y}^2(s) + \dot{y}^2(s))\, ds. \tag{4.9}$$

For system (4.6a) its scalar equation in z is the same as (4.9) with y replaced by z.

For system (4.4), let $u = y \mp z$. Then $u' = z$, $z'' = \pm y^2 \pm A = \pm(z \pm u)^2 \pm A$, we have $u''' = \pm(u' \pm u)^2 \pm A$, multiply by $(u' \pm u)'$ to get

$$u'''(u' \pm u)' = \pm(u' \pm u)^2(u' \pm u)' \pm A(u' \pm u)'$$

or

$$\pm u''^2 + 2u'u'' \mp \frac{2}{3}(u' \pm u)^3 - 2Au' \mp 2Au = C + 2\int_0^t u''^2(s)\, ds \tag{4.10}$$

where $A \neq 0$ and C is an arbitrary constant. Since hypotheses $(H1)$ and $(H2)$ are satisfied, theorem 2.1 implies that system (4.1a), (4.6a) and (4.4) are not chaotic. The following systems can be integrated to:

$$\text{System (4.2)} \quad \ddot{y} - y^2 \mp y = C \mp At$$

$$\text{System (4.3)}\ y = z + At + C, \quad \ddot{z} - z^2 \mp z = \pm At \pm C$$

$$\text{System (4.5)} \quad \ddot{y} - y^2 = \pm At + C$$

$$\text{System (4.8)} \quad \ddot{z} - \frac{1}{2}z^2 \mp z = C + At$$

where $A \neq 0$ and C are arbitrary constant. Assume that all the solutions of the four systems are bounded for all $t \geqslant 0$. Since the left hand sides of (4.2), (4.3), (4.5), and (4.8) are polynomials and so they are bounded. But their right-hand sides go to infinity as $t \to \infty$, a contradiction. Hence system (4.2), (4.3), (4.5), and (4.8) are not chaotic. The proof is completed. □

Remark 4.2 *System* (4.6b) *has scalar equation* $\dddot{z} = z^2 - \dot{z} + A$. *Clearly it is equivalent to* (4.1b).

Consider system (4.7)

$$\begin{cases} x' = yz + A, & A \neq 0,\ t \geqslant 0 \\ y' = x \pm z, & z' = x \end{cases}$$

Lemma 4.3 *If* $A > 0$, *then system* (4.7) *is not chaotic.*

Proof: For (4.7), let $u = y - z$. Then $u' = \pm z$, $z'' = yz + A = (u + z)z + A$. Therefore we have $u''' = (u \pm u')u' \pm A$. Integrate it to get

$$u'' - \frac{1}{2}u^2 = C \pm At \pm \int_0^t u'^2(s)\, ds$$

where $A > 0$ and C is arbitrary constant. For any bounded solution for $t \geqslant 0$, the left-hand side stays bounded for $t \geqslant 0$, but the right-hand side goes to infinity as t goes to infinity, a contradiction. Hence systems (4.7) is not chaotic. □

Our study of system (4.7) for $A < 0$ indicates that it is a little more complicated than other systems except system (4.1b) due to its oscillatory pattern. We can not prove analytically if (4.7) when $A < 0$ is chaotic, but we show in the following two theorems that there are positively invariant regions in both (4.7a) and (4.7b) and if a solution of (4.7a) doesn't reach the invariant region for all $t \geqslant 0$, it is either oscillatory or $y(\infty) = \pm\infty$ with $x(\infty) = z(\infty) = 0$ and if a solution of (4.7b) doesn't reach the invariant region for all $t \geqslant 0$, then it is either oscillatory or $y(\infty) = 1$ with $x(\infty) = \infty(-\infty)$ and $z(\infty) = \infty(-\infty)$.

Let $p(t) = (x(t),\, y(t),\, z(t))$ denote solutions of (4.7a) and $p_0 = p(0)$ be the initial values. For (4.7a), i.e. the " + " system of (4.7), we define the 3D set

$$\bar{\Omega} := \{(x,\, y,\, z)|\, yz + A \geqslant 0,\ y > 0,\ x \geqslant 0,\ A < 0\}$$

where $\partial\Omega$ denotes the boundary of $\bar{\Omega}$, and $\Omega = \bar{\Omega} \diagdown \partial\Omega$.

Theorem 4.4 *For system* (4.7a) *with* $A < 0$*, if there exists a* $T > 0$ *such that* $p(T) \in \bar{\Omega}$*, then* $p(t) \in \bar{\Omega}$ *for all* $t \geqslant T$ *and* $p(t) \to (\infty,\, \infty,\, \infty)$ *as* $t \to \infty$*. Otherwise the solutions are either oscillatory or satisfy*

$$y(t) \to \pm\infty \ \textit{and}\ x(t) \to 0\ \textit{and}\ z(t) \to 0,\ \textit{as} \quad t \to \infty. \tag{4.11}$$

Proof: Let K_i, $i = 1, 2, 3$ be constants. We assume that $x(\infty)$, $y(\infty)$, and $z(\infty)$ exist. That is one of the 27 cases $x(\infty) = K_1, \infty$ or $-\infty$, $y(\infty) = K_2, \infty$ or $-\infty$, and $z(\infty) = K_3, \infty$ or $-\infty$ will happen. We show that the only possible limits are as in (4.11) or $p(\infty) = (\infty,\, \infty,\, \infty)$. Then we show that if there exists a $T > 0$ such that $p(T) \in \bar{\Omega}$, then $p(t) \in \bar{\Omega}$ for all $t \geqslant T$ and $p(\infty) = (\infty,\, \infty,\, \infty)$. For the 27 cases, it is sufficient to consider the following cases as $t \to \infty$:
Case 1. $(x, y, z) \to (K_1, K_2, K_3)$.
Since the system has no fixed point, this is impossible.
Case 2. $x \to K_1$ (or $|x| \to \infty$), $y \to K_2$ and $|z| \to \infty$ (or $z \to K_3$).
Since $|y'| = |x + z| \to \infty$, $|y| \to \infty$, a contradiction.
Case 3. $x \to K_1$, $y \to \infty(-\infty)$ and $z \to K_3$.
If $y \to \infty$ and $K_3 > 0$ ($K_3 < 0$), then $x' = yz + A \to \infty(-\infty)$, and therefore $|x| \to \infty$, a contradiction. If $K_3 = 0$, since $z' = x \to K_1$, $z' \to K_1 = 0$. From $y' = x + z \to 0$ one obtains

$y'' \to 0$ and $y''' \to 0$. Since $y'''(t) = y'(t)z(t) + y(t)y'(t) + A \to 0$, $y(t)y'(t) \to -A$. This leaves (4.11) possible.

If $y(\infty) = -\infty$, a similar argument applies.

Case 4. $(x, |y|, |z|) \to (K_1, \infty, \infty)$.

Then $|x'| = |yz + A| \to \infty$ which contradicts that $x \to K_1$.

Case 5. $x \to \infty(-\infty)$, $y \to K_2$, and $z \to \infty(-\infty)$ or $x \to \infty(-\infty)$, $y \to -\infty(\infty)$, and $z \to K_3$.

Then $|y'| = |x+z| \to \infty$ which contradicts that $y \to K_2$. For the second case, $y' = x+z \to \pm\infty$ contradicts that $y \to \mp\infty$.

Case 6. $x \to \infty(-\infty)$, $|y| \to \infty$ (or $|K_2|$), and $z \to -\infty(\infty)$ or $x \to \infty(-\infty)$, $y \to \infty(-\infty)$, and $z \to K_3$.

Then $z' = x \to \pm\infty$ which contradicts that $z \to \mp\infty$(or K_3).

Case 7. $x \to \infty(-\infty)$, $y \to -\infty(\infty)$, and $z \to \infty(-\infty)$.

Then $y' = x + z \to \infty(-\infty)$ which contradicts that $y \to -\infty(\infty)$.

Case 8. $(x, y, z) \to (-\infty, -\infty, -\infty)$.

Then $x' = yz + A \to \infty$ which contradicts that $x \to -\infty$.

Up to now, we have proved that the solutions of the system do not go to a limit except in the two cases: a) $p(t) \to (0, \pm\infty, 0)$, b) $p(t) \to (\infty, \infty, \infty)$. Since

$$\partial\Omega := \{(x,\, y,\, z) |\, yz + A = 0,\; y \geqslant 0,\; x \geqslant 0, \quad \text{and} \quad yz + A > 0,\; y \geqslant 0,\; x = 0,\; A < 0\}$$

$\partial\Omega = S_4 \cup S_5 \cup \Gamma$, where $S_4 = \{(x,\, y,\, z) \,|\, yz+A = 0,\; y \geqslant 0,\; x > 0,\; A < 0\}$, $S_5 = \{(x,\, y,\, z) \,|\, yz + A > 0,\; y \geqslant 0,\; x = 0,\; A < 0\}$, and $\Gamma = \{(x,\, y,\, z) \,|\, yz + A = 0,\; y \geqslant 0,\; x = 0,\; A < 0\}$.

Suppose that $p(T) \in \partial\Omega$.

If $p(T) \in S_4$, then at $t = T$, $x > 0$, $y > 0$, $z > 0$, $x' = zy+A = 0$, and so $x'' = yx+(x+z)z > 0$. This implies that $x(T)$ is a local minimum. Since $y'(T) = x(T)+z(T) > 0$ and $z'(T) = x(T) > 0$, $\exists\, \eta > 0$ such that $x(t) > 0$, $y(t) > 0$, $z(t) > 0$, and $x'(t) = z(t)y(t)+A > 0$ for $t \in (T,\, T+\eta)$, i.e. $p(t) \in \Omega \diagdown \partial\Omega$.

If $p(T) \in S_5$, then at $t = T$, $x = 0$, $y > 0$, $z > 0$, and $x' = zy + A > 0$, and so $y' = x + z > 0$, $z' = x = 0$, $z'' = x' > 0$. This implies that $z(T)$ is a local minimum. Therefore there exists an $\eta > 0$ such that $x(t) > 0$, $y(t) > 0$, $z(t) > 0$, and $x'(t) = z(t)y(t) + A > 0$, i.e. $p(t) \in \Omega \diagdown \partial\Omega$, for $t \in (T,\, T + \eta)$.

If $p(T) \in \Gamma$, then at $t = T$, $x = 0$, $y > 0$, $z > 0$, and $x' = zy + A = 0$, and so $x'' = yx + (x + z)z = z^2 > 0$. This implies that $x(T)$ is a local minimum. $y'(T) = x(T) + z(T) > 0$, $z'(T) = x(T) = 0$, $z''(T) = x'(T) = 0$, $z'''(T) = x''(T) = z^2(T) > 0$ imply that $z(T)$ and $z'(T)$ are local minima. Therefore there exists an $\eta > 0$ such that $x(t) > 0$, $y(t) > 0$, $z(t) > 0$, and $x'(t) = z(t)y(t) + A > 0$, i.e. $p(t) \in \Omega \diagdown \partial\Omega$, for $t \in (T,\, T + \eta)$.

Since for $p(t) \in \bar{\Omega}$ $x' \geqslant 0$, $y' \geqslant 0$ and $z' \geqslant 0$, the solutions have limits and it is easy to see that $p(\infty) = (\infty,\, \infty,\, \infty)$. This completes the proof of the theorem. □

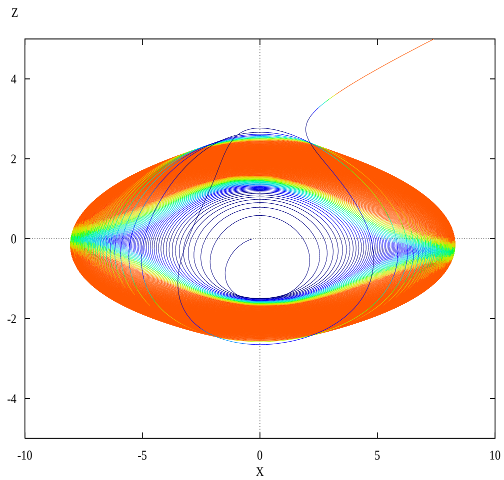

Figure 8: (4.7a)'s phase space, $x(0)$=-0.3571, $y(0)$ =-1.1628, $z(0)$ =-0.0125, A=-2.04

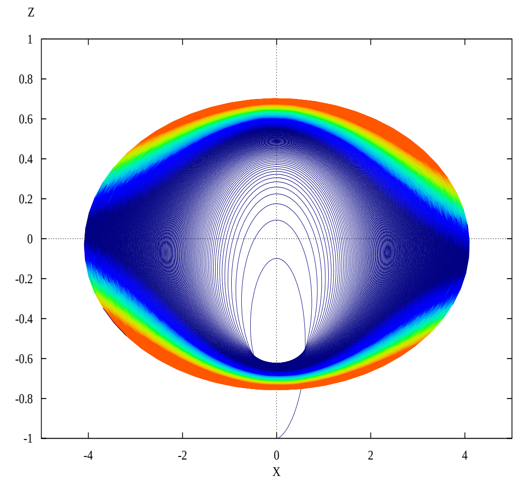

Figure 9: (4.7a)'s phase space, $x(0)$ =-0.01 $y(0)$ =-1.1628, $z(0)$ =-1, A=-1.04

For the unresolved cases in system (4.7a), numerical simulations indicate that there can't be more complicated solutions than the ones shown in figure 14 and figure 15. The solutions can be oscillatory at the beginning. They will eventually either go to a limit or approach a periodic orbit. This suggests that the system has no chaos.

For (4.7b), ie. the " $-$ " system, we define the set

$$\bar{\Omega}_1 := \{(x,\, y,\, z)|\, z(y-1)+A \geqslant 0,\, z>0,\, x-z \geqslant 0,\, A<0\}$$

where $\partial\Omega_1$ denotes the boundary of $\bar{\Omega}_1$, and $\Omega_1 = \bar{\Omega}_1 \diagdown \partial\Omega_1$.

Theorem 4.5 *For system* (4.7b) *with* $A<0$, *if there exists a* $T>t_0$ *such that* $p(T) \in \bar{\Omega}_1$, *then* $p(t) \in \bar{\Omega}_1$ *for all* $t \geqslant T$ *and* $p(t)$ *has a limit as* $t \to \infty$. *Otherwise the solutions are either oscillatory or satisfy*

$$y \to 1, x \to \infty(-\infty), \;\; and\; z \to \infty(-\infty), \qquad as\; t \to \infty \tag{4.12}$$

Proof : The proof is similar to that of Theorem 4.4.

Numerical simulations on system (4.7b) supports our claim that the only limit the solutions can go to is the limit in (4.12), see figure 10. Similar to system (4.7a), there can be solutions that approach a periodic orbit, see figure 11. However our numerical results suggest that (4.7b) is not chaotic.

5 The simplest chaotic conservative system

The Logistic map $x_{n+1} = \mu x_n(1-x_n)$ is one of the simplest dynamical systems that exhibit chaos. Despite its simplicity, for most $\mu \in (3.5699, 4]$, the sequence $\{x_n\}$ it generates appears

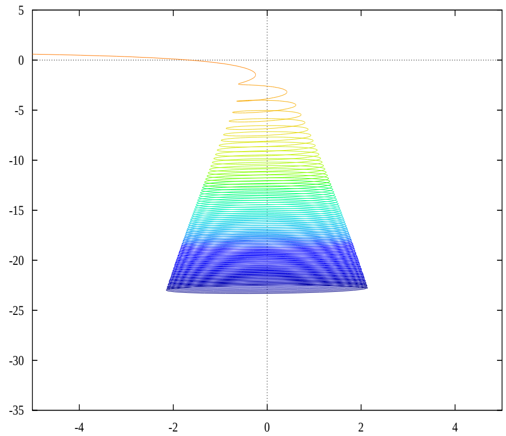

Figure 10: (4.7b)'s phase space, $x(0)$ =2.0975 $y(0)$ =-22.76, $z(0)$ =0.0125, A=-2.04

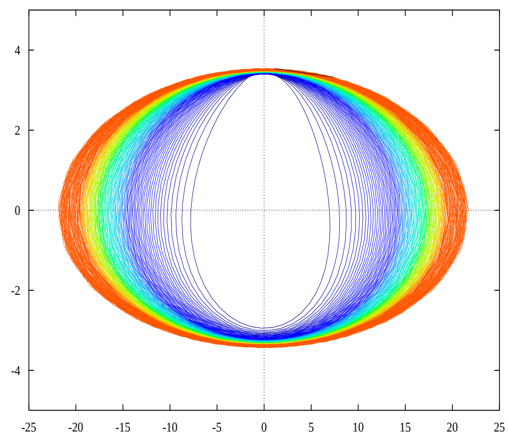

Figure 11: (4.7b)'s phase space, $x(0)$ =0.578 $y(0)$ =-0.76, $z(0)$ =3.466, A=-2.04

to have sensitive dependence on initial conditions or chaos. Actually many features of chaos can be found in logistic map. One definition of chaotic orbit is that it has positive Lyapunov exponent if the orbit is not periodic. The Lyapunov exponent of a smooth map $x_{n+1} = f(x_n)$ at an initial point x_1 is defined as the limit

$$h(x_1) = \lim_{n\to\infty} (1/n)[\ln|f'(x_1)| + ... + \ln|f'(x_n)|]$$

if it exists. For example when $\mu = 4$, there are non-periodic orbits with Lyapunov exponent $\ln 2$ (Proof of Theorem 3.13, Alligood et. al, 1996).

For ordinary differential equations $\dot{x} = f(x)$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous, the solutions can be chaotic only when $n \geqslant 3$ due to Poincaé-Bendixon theorem. We say the solution $x(t)$ is chaotic if (a) it is bounded, (b) it has a positive Lyapunov exponent (c) it is not an equilibrium point, a periodic orbit or asymptotic to an equilibrium point or a periodic orbit. Chaotic dissipative systems usually have strange attractors while chaotic conservative systems do not.

We've shown analytically in this paper and in (Zhang and Heidel, 1997) and (Heidel and Zhang, 1999) that all of the 3D autonomous quadratic systems with four terms on the right hand side and all five term conservative system with one nonlinear term are not chaotic except for the systems

$$x' = y^2 - z^2,\ y' = x,\ z' = y \tag{5.1}$$

(3.11b), (4.7) and (3.8) for certain range of the parameter A and (4.1b). Analytic and numerical studies show that there is no chaos in systems (5.1), (3.11b), (4.7) and (3.8).

Now we consider system (4.1b) $x' = y^2 - z + A$, $y' = z$, $z' = x$, where $A \neq 0$. It has scalar equation

$$y''' = y^2 - y' + A \tag{5.2}$$

Sprott (1997a) used a fourth-order Runge-Kutta integrator with a step-size $\Delta t = 0.05$ to equation (5.2) and showed that negative A values less than 0 and greater than -0.05 produce chaotic solutions for selected initial conditions. In Sprott's words large negative values of $A(\cong -0.05)$ are chaotic in most initial conditions, but the trajectory quickly escapes. As A approaches zero, the range of initial conditions that produce chaos shrink to zero and escape time approaches infinity. For $A = -0.01$, it produces a toroidal structure. A Poincaré section $x = 0$ for 21 different initial conditions are shown in Sprott's (1997) Figure-4. It appears that in the vicinity of the separatrix which surround the period 8, 9, 10, and 11 islands chaos occurs.

Another evidence (Sprott, 2000) that the system has chaos is that it has positive Lyapunov exponent. System (4.1b) is linearly equivalent to Sprott's (2000) system No. 2 (Page 759) $x' = y$, $y' = z$, $z' = By \pm x + x^2$, $(B = -2.8)$:

An affine transformation

$$x = (\sqrt{-B})^3 X \mp \frac{1}{2},\ y = B^2 Y,\ z = (\sqrt{-B})^5 Z,\ t = \frac{1}{\sqrt{-B}} T$$

leads to

$$X' = Y,\ Y' = Z,\ Z' = -Y + X^2 + \frac{1}{4B^3}$$

with scalar equation

$$X''' = X^2 - X' + \frac{1}{4B^3} \tag{5.3}$$

which is in the form of the scalar equation of system (4.1b). For $B = -2.8$ ($A = -0.011388$ in (4.1b)), $(x, x', x'') = (\mp 0.5, -1, 1)$ using a fourth-order Runge-Kutta integrator with a step-size $\Delta t = 0.05$ Sprott (2000) calculated the Lyapunov exponents and they are $(0.002, 0, -0.002)$ base e.

In our numerical simulations of (4.1b) we use $\Delta t = 0.005$ and they are shown in figures 12 to 14. Figure 12 is a trajectory with the parameter $A = -0.0125$. Figure 13 is a Poincaré section at $x = 0$ with 10 different initial conditions $0.01 \leqslant x \leqslant 0.12$, $y = -0.105714$, $z = -0.102325$. Most of them are quasi-periodic orbits and they are attracted to 2D tori. From figure 14 it appears that the trajectory fills a 3D space. But as $t = 24658.66$, $x \cong 0$, $y = 0.1742977$, $z \cong 0$ the calculation stopped. We think that our simulations confirm Sprott's result. Thus our study corroborates Sprott that this system is the algebraically simplest conservative chaotic system one can find.

The following 5-2 system (Sprott, 1994 Case A) is a less simple conservative chaotic system. Sprott (1994) discovered a 5-2 conservative system

$$x' = y,\ y' = -x + yz,\ z' = 1 - y^2 \tag{5.4}$$

with chaotic solutions. It has Lyapunov exponents (0.014, 0, -0.014) and therefore Lyapunov dimension $D_L = 3$. A chaotic solution with initial condition (0, 5, 0) can be obtained. Since the sum of the three exponents is the average rate of fractional volume expansion along the

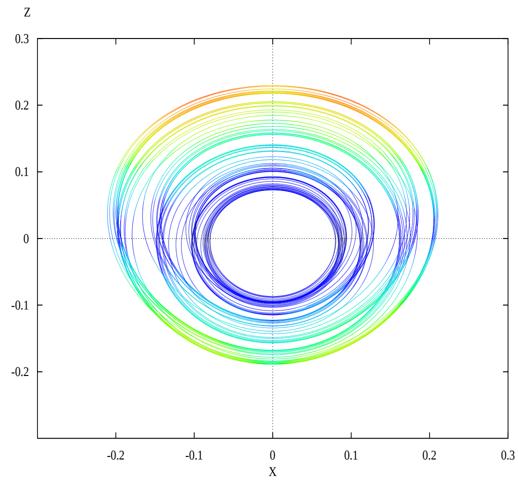

Figure 12: (4.1b)'s "chaotic" orbit, $x(0) = 0.0428571$, $y(0) = -0.105714$, $z(0) = -0.102325$, $A = -0.0125$

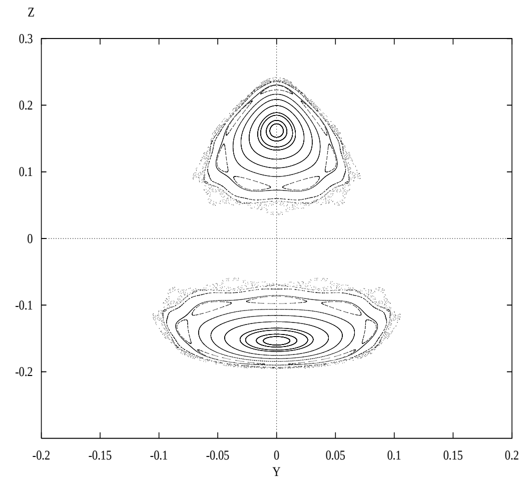

Figure 13: (4.1b)'s Poincaré section at $x = 0$ with 10 different initial conditions $0.01 \leqslant x \leqslant 0.12$, $y = -0.105714$, $z = -0.102325$ and $A = -0.0125$

trajectory, it is a conservative system. A Poincaré section at $z = 0$ of 37 solutions with initial conditions $-2.5 \leqslant x(0) \leqslant 2.5$, $1.25 \leqslant y(0) \leqslant 5.25$, $z(0) = 0$ of system (5.4) is shown in figure 15.

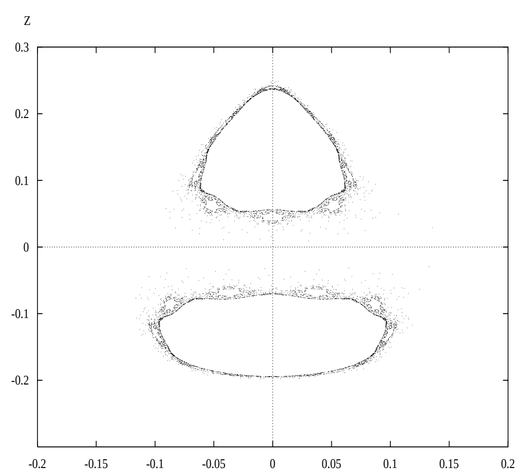

Figure 14: (4.1b)'s Poincaré section at $x = 0$ with initial condition $x(0) = 0.01$, $y(0) = -0.105714$, $z(0) = -0.102325$, $A = -0.0125$

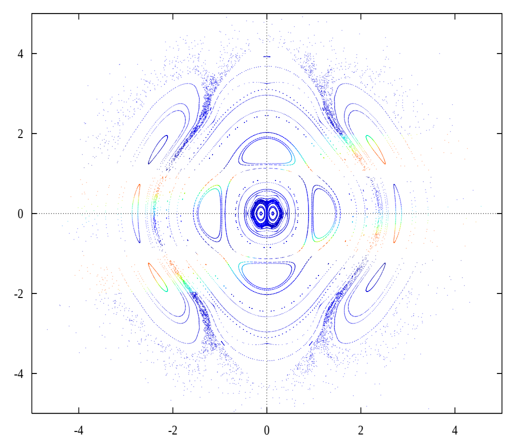

Figure 15: Poincaré section of (5.4) at $z = 0$ with 37 different initial conditions $-2.5 \leqslant x(0) \leqslant 2.5$, $1.25 \leqslant y(0) \leqslant 5.25$, $z(0) = 0$

Toland (1988) studied the following problem

$$\lambda u''' + u' = 1 - u^2, \quad \tau \in R, \ \lambda < 0 \tag{5.5}$$

$$\lim_{\tau \to \pm\infty} u(\tau) = \pm 1, \quad ' = \frac{d}{d\tau}$$

Letting $w = v'$, $v = u'$, (5.5) becomes the system:

$$\begin{cases} w' = -\frac{1}{\lambda}u^2 - \frac{1}{\lambda}v + \frac{1}{\lambda} \\ u' = v, \quad v' = w, \end{cases} \tag{5.6}$$

We rescale system (5.6), and for $\lambda > 0$, let $w = \mp\lambda^{-\frac{3}{2}}x$, $u = \mp\lambda^{-\frac{1}{2}}y$, $w = \mp\lambda^{-1}z$, $\tau = \pm\lambda^{\frac{1}{2}}t$, and $A = -\lambda$, and then (5.6) becomes (4.1b). Toland (1988) showed that for $\lambda < 0(A > 0$ in (4.1b)) on an open interval, there exists a unique monotone heteroclinic orbit. Jones and Troy (1992) studied the 3rd order equation

$$w''' + w' = c^2 - w^2/2, \quad \tau \in (-\infty, \infty) \text{ and } ' = \frac{d}{d\tau} \tag{5.7}$$

where $c > 0$ is a constant, as a steady solutions of the Kuramoto-Sivashinsky equation for small wave speed. Equation (5.7) is the scalar equation of the 3D system

$$\begin{cases} v' = c^2 - w^2/2 - u \\ w' = u, \quad u' = v, \end{cases} \tag{5.8}$$

and system (4.1b) can be transformed to system (5.8) by the transformation

$$x = -\frac{1}{2}v, \quad y = -\frac{1}{2}w, \quad z = -\frac{1}{2}u, \quad t = \tau, \quad c^2 = -2A, \ A < 0$$

One of the results they obtained for system (5.7) is

Theorem 5.1 *There exists $\bar{c} > 0$ such that for each $c \in (0, \bar{c})$ there is an odd periodic solution of* (5.7)*.*

In the following, we prove that system (4.1b) has no chaos for all $A > 0$.

Proposition 5.2 *If $A > 0$, then system* (4.1b) *is not chaotic.*

Proof. From (4.1b), we have $y''' = y^2 - y' + A$ or

$$y'' + y = C + At + \int_0^t y^2(s)\, ds$$

C is an arbitrary constant. For $A > 0$, assume that all the solutions are bounded for all $t \geqslant 0$. Therefore $y'' + y$ is bounded for all $t \geqslant 0$. But $y''(\infty) + y(\infty) = \infty$, a contradiction. Hence system (4.1b) with $A > 0$ is not chaotic. $\square$

6 Appendix. Equivalent systems

Each of the following 5-1 conservative systems is linearly equivalent to a 5-1 or 4-1 system given by proposition 6.2:

$$x' = y^2 + y + Az, \quad y' = z, \quad z' = \pm x \tag{6.1}$$

$$x' = y^2 + Ay, \quad y' = x \pm z, \quad z' = x \tag{6.2}$$

$$x' = yz + Az, \quad y' = x \pm z, \quad z' = x \tag{6.3}$$

$$x' = Ay^2 + y + 1, \quad y' = \pm z, \quad z' = x \tag{6.4}$$

$$x' = y^2 \pm z + A, \quad y' = x, \quad z' = y \tag{6.5}$$

$$x' = Ay^2 + y, \quad y' = \pm(z + 1), \quad z' = x \tag{6.6}$$

$$x' = y^2 \pm z, \quad y' = x + A, \quad z' = y \tag{6.7}$$

$$x' = y^2 \pm z, \quad y' = z + A, \quad z' = x \tag{6.8}$$

$$x' = yz \pm y + A, \quad y' = x, \quad z' = y \tag{6.9}$$

$$x' = yz \pm y + A, \quad y' = z, \quad z' = x \tag{6.10}$$

$$x' = yz \pm y, \quad y' = x + A, \quad z' = x \tag{6.11}$$

$$x' = yz \pm y, \quad y' = z + A, \quad z' = x \tag{6.12}$$

$$x' = z^2 + z, \quad y' = x + A, \quad z' = \pm y \tag{6.13}$$

$$x' = z^2 \pm y, \quad y' = x + A, \quad z' = y \tag{6.14}$$

$$x' = z^2 + y, \quad y' = \pm z + A, \quad z' = x \tag{6.15}$$

$$x' = yz \pm z, \quad y' = x + A, \quad z' = x \tag{6.16}$$

$$x' = yz \pm z, \quad y' = x + A, \quad z' = y \tag{6.17}$$

$$x' = yz \pm z, \quad y' = z + A, \quad z' = x \tag{6.18}$$

$$x' = y^2, \quad y' = x \pm z + A, \quad z' = x \tag{6.19}$$

$$x' = z^2, \quad y' = x \pm z + A, \quad z' = y \tag{6.20}$$

$$x' = yz, \quad y' = \pm x + z + A, \quad z' = x \tag{6.21}$$

$$x' = y^2, \quad y' = x \pm z, \quad z' = x + A \tag{6.22}$$

$$x' = y^2 + A, \quad y' = \pm z, \quad z' = x + B \tag{6.23}$$

$$x' = z^2, \quad y' = x \pm z, \quad z' = y + A \tag{6.24}$$

$$x' = yz, \quad y' = \pm x + z, \quad z' = x + A \tag{6.25}$$

$$x' = yz, \quad y' = x \pm z, \quad z' = y + A \tag{6.26}$$

Remark 6.1 *The above 26 systems are either equivalent to* (3.1)-(3.11) *and* (4.1)- (4.8) *or equivalent to 4-term equations.*

The notation "$\backsim$" represents "be equivalent to" under linear transformation.

Proposition 6.2 *The following linearly equivalent relations hold:*

(3.1) $\backsim$ (6.15); (3.8) $\backsim$ (6.21) $\backsim$ (6.25); (3.9) $\backsim$ (6.26);

(4.1) $\backsim$ (6.1) $\backsim$ (6.8); (4.4) $\backsim$ (6.2); (6.4) $\backsim$ (6.6) $\backsim$ (6.13);

(6.5) $\backsim$ (6.7); (6.9) $\backsim$ (6.10); (6.11) $\backsim$ (6.16);

(6.12) $\backsim$ (6.17) $\backsim$ (6.18); (6.19) $\backsim$ (6.22); (6.20) $\backsim$ (6.24);

Systems (6.3), (6.4), (6.5), (6.9), (6.11), (6.12), (6.14), (6.19), (6.20) *and* (6.23) *are equivalent to 4 term conservative systems and therefore they are not chaotic.*

Proof: The proof is straight forward. □

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