Chapter 6

SOLUTION OF VISCOUS-FLOW PROBLEMS

6.1 Introduction

THE previous chapter contained derivations of the relationships for the conservation of mass and momentum—the *equations of motion*—in rectangular, cylindrical, and spherical coordinates. All the experimental evidence indicates that these are indeed the most fundamental equations of fluid mechanics, and that in principle they govern any situation involving the flow of a Newtonian fluid. Unfortunately, because of their all-embracing quality, their solution in analytical terms is difficult or impossible except for relatively simple situations. However, it is important to be aware of these "Navier-Stokes equations," for the following reasons:

1. They lead to the analytical and exact solution of some simple, yet important problems, as will be demonstrated by examples in this chapter.
2. They form the basis for further work in other areas of chemical engineering.
3. If a few realistic *simplifying assumptions* are made, they can often lead to approximate solutions that are eminently acceptable for many engineering purposes. Representative examples occur in the study of boundary layers, waves, lubrication, coating of substrates with films, and inviscid (irrotational) flow.
4. With the aid of more sophisticated techniques, such as those involving power series and asymptotic expansions, and *particularly* computer-implemented numerical methods, they can lead to the solution of moderately or highly advanced problems, such as those involving injection-molding of polymers and even the incredibly difficult problem of weather prediction.

The following sections present exact solutions of the equations of motion for several relatively simple problems in rectangular, cylindrical, and spherical coordinates. Throughout, unless otherwise stated, the flow is assumed to be *steady*, *laminar* and *Newtonian,* with *constant* density and viscosity. Although these assumptions are necessary in order to obtain solutions, they are nevertheless realistic in many cases.

All of the examples in this chapter are characterized by low Reynolds numbers. That is, the viscous forces are much more important than the inertial forces, and are usually counterbalanced by pressure or gravitational effects. Typical applications occur at low flow rates and in the flow of high-viscosity polymers. Situations in which viscous effects are relatively *unimportant* will be discussed in Chapter 7.

Solution procedure. The general *procedure* for solving each problem involves the following steps:

1. Make reasonable simplifying assumptions. Almost all of the cases treated here will involve *steady incompressible* flow of a *Newtonian* fluid in a *single* coordinate direction. Further, *gravity* may or may not be important, and a certain amount of *symmetry* may be apparent.
2. Write down the *equations of motion*—both mass (continuity) and momentum balances—and simplify them according to the assumptions made previously, striking out terms that are zero. Typically, only a very few terms—perhaps only one in some cases—will remain in each differential equation. The simplified continuity equation usually yields information that can subsequently be used to simplify the momentum equations.
3. *Integrate* the simplified equations in order to obtain expressions for the dependent variables such as velocities and pressure. These expressions will usually contain some, as yet, arbitrary constants—typically *two* for the velocities (since they appear in second-order derivatives in the momentum equations) and *one* for the pressure (since it appears only in a first-order derivative).
4. Invoke the *boundary conditions* in order to evaluate the constants appearing in the previous step. For *pressure,* such a condition usually amounts to a specified pressure at a certain location—at the inlet of a pipe, or at a free surface exposed to the atmosphere, for example. For the *velocities*, these conditions fall into either of the following classifications:
   (a) Continuity of the velocity, amounting to a *no-slip* condition. Thus, the velocity of the fluid in contact with a solid surface typically equals the velocity of that surface—zero if the surface is stationary.[1] And, for the few cases in which one fluid (A, say) is in contact with another immiscible fluid (B), the velocity in fluid A equals the velocity in fluid B at the common interface.
   (b) Continuity of the shear stress, usually between two fluids A and B, leading to the product of viscosity and a velocity gradient having the same value at the common interface, whether in fluid A or B. If fluid A is a liquid, and fluid B is a relatively stagnant gas, which—because of its low viscosity—is incapable of sustaining any significant shear stress, then the common shear stress is effectively zero.
5. At this stage, the problem is essentially solved for the pressure and velocities. Finally, if desired, shear-stress distributions can be derived by differentiating

[1] In a few exceptional situations there may be lack of adhesion between the fluid and surface, in which case *slip* can occur.

the velocities in order to obtain the velocity *gradients*; numerical predictions of process variables can also be made.

Types of flow. Two broad classes of viscous flow will be illustrated in this chapter:

1. *Poiseuille flow,* in which an applied pressure difference causes fluid motion between stationary surfaces.
2. *Couette flow,* in which a moving surface drags adjacent fluid along with it and thereby imparts a motion to the rest of the fluid.

Occasionally, it is possible to have both types of motion occurring simultaneously, as in the screw extruder analyzed in Example 6.4.

6.2 Solution of the Equations of Motion in Rectangular Coordinates

The remainder of this chapter consists almost entirely of a series of worked examples, illustrating the above steps for solving viscous-flow problems.

Example 6.1—Flow Between Parallel Plates

Fig. E6.1.1 shows the flow of a fluid of viscosity μ, which flows in the x direction between two rectangular plates, whose width is very large in the z direction when compared to their separation in the y direction. Such a situation could occur in a die when a polymer is being extruded at the exit into a sheet, which is subsequently cooled and solidified. Determine the relationship between the flow rate and the pressure drop between the inlet and exit, together with several other quantities of interest.

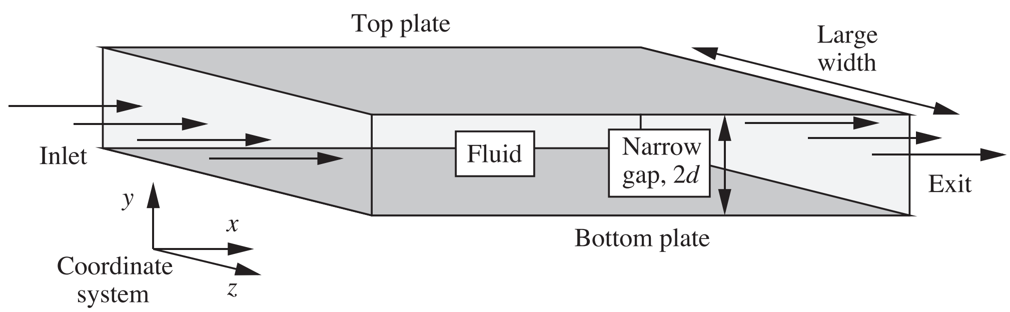


Fig. E6.1.1 Geometry for flow through a rectangular duct. The spacing between the plates is exaggerated in relation to their length.

Simplifying assumptions. The situation is analyzed by referring to a cross section of the duct, shown in Fig. E6.1.2, taken at any fixed value of z. Let the depth be $2d$ ($\pm d$ above and below the centerline or axis of symmetry $y = 0$), and the length L. Note that the motion is of the *Poiseuille* type, since it is caused by the applied pressure difference $(p_1 - p_2)$. Make the following realistic *assumptions* about the flow:

1. As already stated, it is steady and Newtonian, with constant density and viscosity. (These assumptions will often be taken for granted, and not restated, in later problems.)
2. There is only one nonzero velocity component—that in the direction of flow, v_x. Thus, $v_y = v_z = 0$.
3. Since, in comparison with their spacing, $2d$, the plates extend for a very long distance in the z direction, all locations in this direction appear essentially identical to one another. In particular, there is no variation of the velocity in the z direction, so that $\partial v_x/\partial z = 0$.
4. Gravity acts vertically downwards; hence, $g_y = -g$ and $g_x = g_z = 0$.
5. The velocity is zero in contact with the plates, so that $v_x = 0$ at $y = \pm d$.

Continuity. Start by examining the general continuity equation, (5.48):

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} = 0, \tag{5.48}$$

which, in view of the constant-density assumption, simplifies to Eqn. (5.52):

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0. \tag{5.52}$$

But since $v_y = v_z = 0$:

$$\frac{\partial v_x}{\partial x} = 0, \tag{E6.1.1}$$

so v_x is independent of the distance from the inlet, and the velocity profile will appear the same for *all* values of x. Since $\partial v_x/\partial z = 0$ (assumption 3), it follows that $v_x = v_x(y)$ is a function of y only.

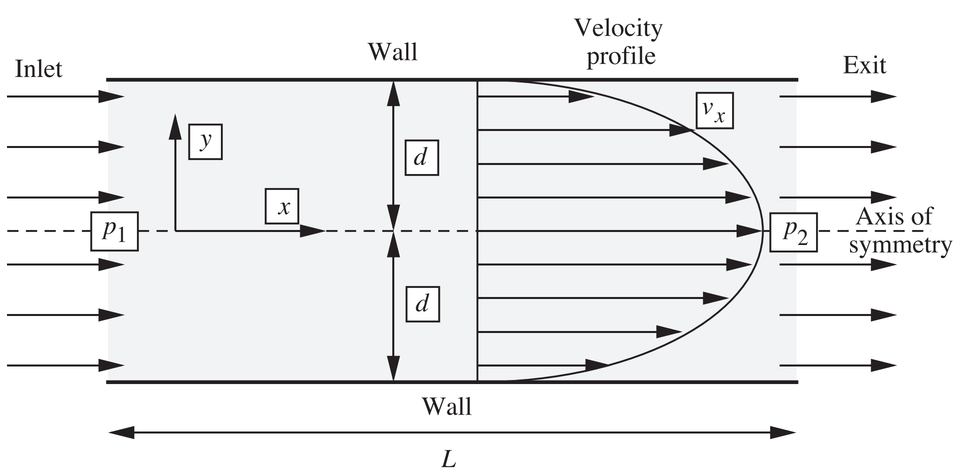


Fig. E6.1.2 Geometry for flow through a rectangular duct.

Momentum balances. With the stated assumptions of a Newtonian fluid with constant density and viscosity, Eqn. (5.73) gives the x, y, and z momentum balances:

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) + \rho g_x,$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right) + \rho g_y,$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right) + \rho g_z.$$

With $v_y = v_z = 0$ (from assumption 2), $\partial v_x/\partial x = 0$ [from the simplified continuity equation, (E6.1.1)], $g_y = -g$, $g_x = g_z = 0$ (assumption 4), and steady flow (assumption 1), these momentum balances simplify enormously, to:

$$\mu\frac{\partial^2 v_x}{\partial y^2} = \frac{\partial p}{\partial x}, \tag{E6.1.2}$$

$$\frac{\partial p}{\partial y} = -\rho g, \tag{E6.1.3}$$

$$\frac{\partial p}{\partial z} = 0. \tag{E6.1.4}$$

Pressure distribution. The last of the simplified momentum balances, Eqn. (E6.1.4), indicates no variation of the pressure across the width of the system (in the z direction), which is hardly a surprising result. When integrated, the second simplified momentum balance, Eqn. (E6.1.3), predicts that the pressure varies according to:

$$p = -\rho g\int dy + f(x) = -\rho g y + f(x). \tag{E6.1.5}$$

Observe carefully that since a *partial* differential equation is being integrated, we obtain not a constant of integration, but a *function* of integration, $f(x)$.

Assume—to be verified later—that $\partial p/\partial x$ is constant, so that the centerline pressure (at $y = 0$) is given by a linear function of the form:

$$p_{y=0} = a + bx. \tag{E6.1.6}$$

The constants a and b may be determined from the inlet and exit centerline pressures:

$$x = 0: \qquad p = p_1 = a, \tag{E6.1.7}$$

$$x = L: \qquad p = p_2 = a + bL, \tag{E6.1.8}$$

leading to:

$$a = p_1, \qquad b = -\frac{p_1 - p_2}{L}. \tag{E6.1.9}$$

Thus, the centerline pressure declines *linearly* from p_1 at the inlet to p_2 at the exit:

$$f(x) = p_1 - \frac{x}{L}(p_1 - p_2), \tag{E6.1.10}$$

so that the complete pressure distribution is

$$p = p_1 - \frac{x}{L}(p_1 - p_2) - \rho g y. \tag{E6.1.11}$$

That is, the pressure declines linearly, both from the bottom plate to the top plate, and *also* from the inlet to the exit. In the majority of applications, $2d \ll L$, and the relatively small pressure variation in the y direction is usually ignored. Thus, p_1 and p_2, although strictly the *centerline* values, are typically referred to as *the* inlet and exit pressures, respectively.

Velocity profile. Since, from Eqn. (E6.1.1), v_x does not depend on x, $\partial^2 v_x/\partial y^2$ appearing in Eqn. (E6.1.2) becomes a *total* derivative, so this equation can be rewritten as:

$$\mu \frac{d^2 v_x}{dy^2} = \frac{\partial p}{\partial x}, \tag{E6.1.12}$$

which is a second-order ordinary differential equation, in which the pressure gradient will be shown to be *uniform* between the inlet and exit, being given by:

$$-\frac{\partial p}{\partial x} = \frac{p_1 - p_2}{L}. \tag{E6.1.13}$$

[A minus sign is used on the left-hand side, since $\partial p/\partial x$ is negative, thus rendering both sides of Eqn. (E6.1.13) as positive quantities.]

Equation (E6.1.12) can be integrated twice, in turn, to yield an expression for the velocity. After multiplication through by dy, a first integration gives:

$$\int \frac{d^2 v_x}{dy^2}\, dy = \int \frac{d}{dy}\left(\frac{dv_x}{dy}\right) dy = \int \frac{1}{\mu}\left(\frac{\partial p}{\partial x}\right) dy,$$

$$\frac{dv_x}{dy} = \frac{1}{\mu}\left(\frac{\partial p}{\partial x}\right) y + c_1. \tag{E6.1.14}$$

A second integration, of Eqn. (E6.1.14), yields:

$$\int \frac{dv_x}{dy}\, dy = \int \left[\frac{1}{\mu}\left(\frac{\partial p}{\partial x}\right) y + c_1\right] dy,$$

$$v_x = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right) y^2 + c_1 y + c_2. \tag{E6.1.15}$$

The two constants of integration, c_1 and c_2, are determined by invoking the boundary conditions:

$$y = 0: \qquad \frac{dv_x}{dy} = 0, \tag{E6.1.16}$$

$$y = d: \qquad v_x = 0, \tag{E6.1.17}$$

leading to:

$$c_1 = 0, \qquad c_2 = -\frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right) d^2. \tag{E6.1.18}$$

Eqns. (E6.1.15) and (E6.1.18) then furnish the velocity profile:

$$v_x = \frac{1}{2\mu}\left(-\frac{\partial p}{\partial x}\right)(d^2 - y^2), \tag{E6.1.19}$$

in which $-\partial p/\partial x$ and $(d^2 - y^2)$ are both *positive* quantities. The velocity profile is *parabolic* in shape, and is shown in Fig. E6.1.2.

Alternative integration procedure. Observe that we have used *indefinite* integrals in the above solution, and have employed the boundary conditions to determine the constants of integration. An alternative approach would again be to integrate Eqn. (E6.1.12) twice, but now to involve *definite* integrals by inserting the boundary conditions as limits of integration.

Thus, by separating variables, integrating once, and noting from symmetry about the centerline that $dv_x/dy = 0$ at $y = 0$, we obtain:

$$\mu \int_0^{dv_x/dy} d\left(\frac{dv_x}{dy}\right) = \frac{\partial p}{\partial x}\int_0^y dy, \tag{E6.1.20}$$

or:

$$\frac{dv_x}{dy} = \frac{1}{\mu}\left(\frac{\partial p}{\partial x}\right) y. \tag{E6.1.21}$$

A second integration, noting that $v_x = 0$ at $y = d$ (zero velocity in contact with the upper plate—the no-slip condition) yields:

$$\int_0^{v_x} dv_x = \frac{1}{\mu}\left(\frac{\partial p}{\partial x}\right)\int_d^y y\, dy. \tag{E6.1.22}$$

That is:

$$v_x = \frac{1}{2\mu}\left(-\frac{\partial p}{\partial x}\right)(d^2 - y^2), \tag{E6.1.23}$$

in which two minus signs have been introduced into the right-hand side in order to make quantities in both parentheses positive. This result is identical to the earlier

Eqn. (E6.1.19). The student is urged to become familiar with both procedures, before deciding on the one that is individually best suited.

Also, the reader who is troubled by the assumption of symmetry of v_x about the centerline (and by never using the fact that $v_x = 0$ at $y = -d$), should be reassured by an alternative approach, starting from Eqn. (E6.1.15):

$$v_x = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right) y^2 + c_1 y + c_2. \tag{E6.1.24}$$

Application of the *two* boundary conditions, $v_x = 0$ at $y = \pm d$, gives

$$c_1 = 0, \qquad c_2 = -\frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right) d^2, \tag{E6.1.25}$$

leading again to the velocity profile of Eqn. (E6.1.19) *without* the assumption of symmetry.

Volumetric flow rate. Integration of the velocity profile yields an expression for the volumetric flow rate Q per unit width of the system. Observe first that the differential flow rate through an element of depth dy is $dQ = v_x dy$, so that:

$$Q = \int_0^Q dQ = \int_{-d}^{d} v_x\, dy = \int_{-d}^{d} \frac{1}{2\mu}\left(-\frac{\partial p}{\partial x}\right)(d^2 - y^2)\, dy = \frac{2d^3}{3\mu}\left(-\frac{\partial p}{\partial x}\right). \tag{E6.1.26}$$

Since from an overall macroscopic balance Q is constant, it follows that $\partial p/\partial x$ is also constant, independent of distance x; the assumptions made in Eqns. (E6.1.6) and (E6.1.13) are therefore verified. The *mean velocity* is the total flow rate per unit depth:

$$v_{xm} = \frac{Q}{2d} = \frac{d^2}{3\mu}\left(-\frac{\partial p}{\partial x}\right), \tag{E6.1.27}$$

and is therefore two-thirds of the maximum velocity, $v_{x\max}$, which occurs at the centerline, $y = 0$.

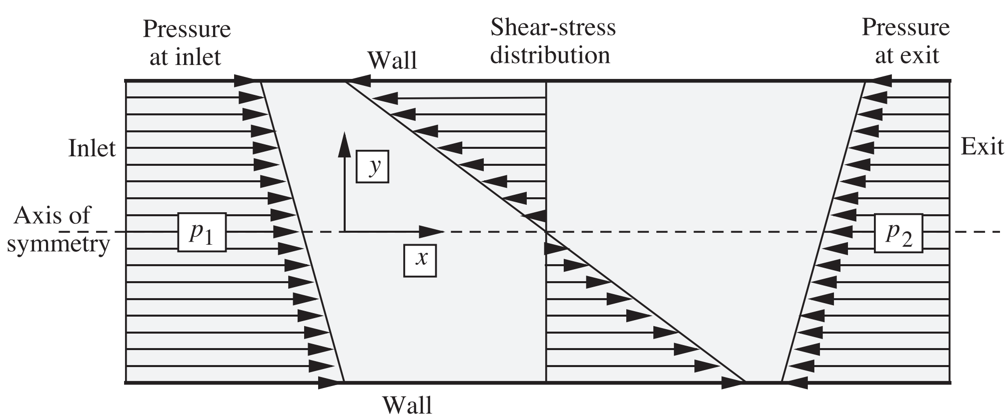


Fig. E6.1.3 Pressure and shear-stress distributions.

Shear-stress distribution. Finally, the shear-stress distribution is obtained by employing Eqn. (5.60):

$$\tau_{yx} = \mu \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right). \tag{5.60}$$

By substituting for v_x from Eqn. (E6.1.15) and recognizing that $v_y = 0$, the shear stress is:

$$\tau_{yx} = -y \left(-\frac{\partial p}{\partial x} \right). \tag{E6.1.28}$$

Referring back to the sign convention expressed in Fig. 5.13, the first minus sign in Eqn. (E6.1.28) indicates for positive y that the fluid in the region of greater y is acting on the region of lesser y in the *negative* x direction, thus trying to retard the fluid between it and the centerline, and acting against the pressure gradient. Representative distributions of pressure and shear stress, from Eqns. (E6.1.11) and (E6.1.28), are sketched in Fig. E6.1.3. More precisely, the arrows at the left and right show the external pressure forces acting on the fluid contained between $x = 0$ and $x = L$. □

6.3 Alternative Solution Using a Shell Balance

Because the flow between parallel plates was the first problem to be examined, the analysis in Example 6.1 was purposely very thorough, extracting the last "ounce" of information. In many other applications, the velocity profile and the flow rate may be the only quantities of prime importance. On the average, therefore, subsequent examples in this chapter will be shorter, concentrating on certain features and ignoring others.

The problem of Example 6.1 was solved by starting with the completely general equations of motion and then simplifying them. An alternative approach involves a direct momentum balance on a differential element of fluid—a "shell"—as illustrated in Example 6.2.

Example 6.2—Shell Balance for Flow Between Parallel Plates

Employ the shell-balance approach to solve the same problem that was studied in Example 6.1.

Assumptions. The necessary "shell" is in reality a differential element of fluid, as shown in Fig. E6.2. The element, which has dimensions of dx and dy in the plane of the diagram, extends for a depth of dz (any other length may be taken) normal to the plane of the diagram.

If, for the present, the element is taken to be a system that is fixed in space, there are three different types of rate of x-momentum transfer to it:

1. A convective transfer of $\rho v_x^2\, dy\, dz$ *in* through the left-hand face, and an identical amount *out* through the right-hand face. Note here that we have implicitly assumed the consequences of the continuity equation, expressed in Eqn. (E6.1.1), that v_x is constant along the duct.
2. Pressure forces on the left- and right-hand faces. The latter will be smaller, because $\partial p/\partial x$ is negative in reality.
3. Shear stresses on the lower and upper faces. Observe that the directions of the arrows conform strictly to the sign convention established in Section 5.6.

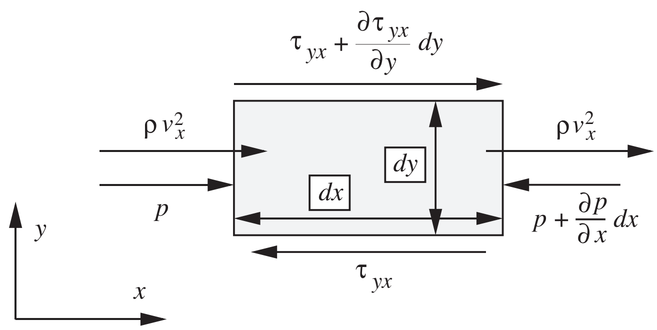


Fig. E6.2 Momentum balance on a fluid element.

A momentum balance on the element, which is not accelerating, gives:

$$\underbrace{\underbrace{\rho v_x^2\, dy\, dz}_{\text{In}} - \underbrace{\rho v_x^2\, dy\, dz}_{\text{Out}}}_{\text{Net convective transfer}} + \underbrace{\underbrace{p\, dy\, dz}_{\text{Left}} - \underbrace{\left(p + \frac{\partial p}{\partial x}\, dx\right) dy\, dz}_{\text{Right}}}_{\text{Net pressure force}}$$

$$+ \underbrace{\underbrace{\left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\, dy\right) dx\, dz}_{\text{Upper}} - \underbrace{\tau_{yx}\, dx\, dz}_{\text{Lower}}}_{\text{Net shear force}} = 0. \quad \text{(E6.2.1)}$$

The usual cancellations can be made, resulting in:

$$\frac{d\tau_{yx}}{dy} = \frac{\partial p}{\partial x}, \quad \text{(E6.2.2)}$$

in which the total derivative recognizes that the shear stress depends only on y and not on x. Substitution for τ_{yx} from Eqn. (5.60) with $v_y = 0$ gives:

$$\mu \frac{d^2 v_x}{dy^2} = \frac{\partial p}{\partial x}, \quad \text{(E6.2.3)}$$

which is identical with Eqn. (E6.1.12) that was derived from the simplified Navier-Stokes equations. The remainder of the development then proceeds as in the previous example. Note that the convective terms can be sidestepped entirely if the momentum balance is performed on an element that is chosen to be *moving* with the fluid, in which case there is no flow either into or out of it. □

The choice of approach—simplifying the full equations of motion, or performing a shell balance—is very much a personal one, and we have generally opted for the former. The application of the Navier-Stokes equations, which are admittedly rather complicated, has the advantages of not "reinventing the (momentum balance) wheel" for each problem, and also of assuring us that no terms are omitted. Conversely, a shell balance has the merits of relative simplicity, although it may be quite difficult to perform convincingly for an element with *curved* sides, as would occur for the problem in spherical coordinates discussed in Example 6.6.

This section concludes with another example problem, which illustrates the application of two further boundary conditions for a liquid, one involving it in contact with a *moving* surface, and the other at a gas/liquid interface where there is a condition of *zero shear.*

Example 6.3—Film Flow on a Moving Substrate

Fig. E6.3.1 shows a coating experiment involving a flat photographic film that is being pulled up from a processing bath by rollers with a steady velocity U at an angle θ to the horizontal. As the film leaves the bath, it entrains some liquid, and in this particular experiment it has reached the stage where: (a) the velocity of the liquid in contact with the film is $v_x = U$ at $y = 0$, (b) the thickness of the liquid is constant at a value δ, and (c) there is no net flow of liquid (as much is being pulled up by the film as is falling back by gravity). (Clearly, if the film were to retain a permanent coating, a net upwards flow of liquid would be needed.)

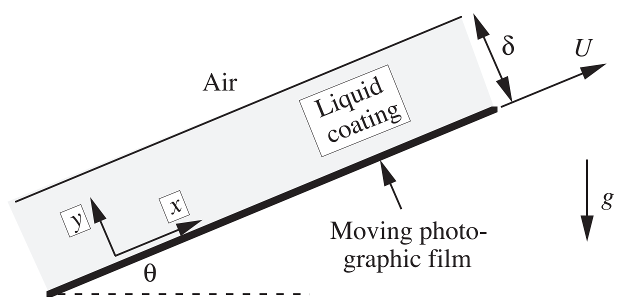


Fig. E6.3.1 Liquid coating on a photographic film.

Perform the following tasks:

1. Write down the differential mass balance and simplify it.
2. Write down the differential momentum balances in the x and y directions. What are the values of g_x and g_y in terms of g and θ? Simplify the momentum balances as much as possible.
3. From the simplified y momentum balance, derive an expression for the pressure p as a function of y, ρ, δ, g, and θ, and hence demonstrate that $\partial p/\partial x = 0$. Assume that the pressure in the surrounding air is zero everywhere.
4. From the simplified x momentum balance, assuming that the air exerts a negligible shear stress τ_{yx} on the surface of the liquid at $y = \delta$, derive an

expression for the liquid velocity v_x as a function of U, y, δ, and α, where $\alpha = \rho g \sin\theta/\mu$.

5. Also derive an expression for the net liquid flow rate Q (per unit width, normal to the plane of Fig. E6.3.1) in terms of U, δ, and α. Noting that $Q = 0$, obtain an expression for the film thickness δ in terms of U and α.
6. Sketch the velocity profile v_x, labeling all important features.

Assumptions and continuity. The following assumptions are reasonable:

1. The flow is steady and Newtonian, with constant density ρ and viscosity μ.
2. The z direction, normal to the plane of the diagram, may be disregarded entirely. Thus, not only is v_z zero, but all derivatives with respect to z, such as $\partial v_x/\partial z$, are also zero.
3. There is only one nonzero velocity component, namely, that in the direction of motion of the photographic film, v_x. Thus, $v_y = v_z = 0$.
4. Gravity acts vertically downwards.

Because of the constant-density assumption, the continuity equation, (5.48), simplifies, as before, to:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0. \tag{E6.3.1}$$

But since $v_y = v_z = 0$, it follows that

$$\frac{\partial v_x}{\partial x} = 0. \tag{E6.3.2}$$

so v_x is independent of distance x along the film. Further, v_x does not depend on z (assumption 2); thus, the velocity profile $v_x = v_x(y)$ depends only on y and will appear the same for *all* values of x.

Momentum balances. With the stated assumptions of a Newtonian fluid with constant density and viscosity, Eqn. (5.73) gives the x and y momentum balances:

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) + \rho g_x,$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right) + \rho g_y,$$

Noting that $g_x = -g\sin\theta$ and $g_y = -g\cos\theta$, these momentum balances simplify to:

$$\frac{\partial p}{\partial x} + \rho g \sin\theta = \mu\frac{\partial^2 v_x}{\partial y^2}, \tag{E6.3.3}$$

$$\frac{\partial p}{\partial y} = -\rho g \cos\theta. \tag{E6.3.4}$$

Integration of Eqn. (E6.3.4), between the free surface at $y = \delta$ (where the gauge pressure is zero) and an arbitrary location y (where the pressure is p) gives:

$$\int_0^p dp = -\rho g \cos\theta \int_\delta^y dy + f(x), \tag{E6.3.5}$$

so that:

$$p = \rho g \cos\theta(\delta - y) + f(x). \tag{E6.3.6}$$

Note that since a *partial* differential equation is being integrated, a *function* of integration, $f(x)$, is again introduced. Another way of looking at it is to observe that if Eqn. (E6.3.6) is differentiated with respect to y, we would recover the original equation, (E6.3.4), because $\partial f(x)/\partial y = 0$.

However, since $p = 0$ at $y = \delta$ (the air/liquid interface) for all values of x, the function $f(x)$ must be zero. Hence, the pressure distribution:

$$p = (\delta - y)\rho g \cos\theta, \tag{E6.3.7}$$

shows that p is not a function of x.

In view of this last result, we may now substitute $\partial p/\partial x = 0$ into the x-momentum balance, Eqn. (E6.3.3), which becomes:

$$\frac{d^2 v_x}{dy^2} = \frac{\rho g}{\mu} \sin\theta = \alpha, \tag{E6.3.8}$$

in which the constant α has been introduced to denote $\rho g \sin\theta/\mu$. Observe that the second derivative of the velocity now appears as a *total* derivative, since v_x depends on y only.

A first integration of Eqn. (E6.3.8) with respect to y gives:

$$\frac{dv_x}{dy} = \alpha y + c_1. \tag{E6.3.9}$$

The boundary condition of zero shear stress at the free surface is now invoked:

$$\tau_{yx} = \mu\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right) = \mu\frac{dv_x}{dy} = 0. \tag{E6.3.10}$$

Thus, from Eqns. (E6.3.9) and (E6.3.10) at $y = \delta$, the first constant of integration can be determined:

$$\frac{dv_x}{dy} = \alpha\delta + c_1 = 0, \qquad \text{or} \qquad c_1 = -\alpha\delta. \tag{E6.3.11}$$

A second integration, of Eqn. (E6.3.9) with respect to y, gives:

$$v_x = \alpha\left(\frac{y^2}{2} - y\delta\right) + c_2. \tag{E6.3.12}$$

The second constant of integration, c_2, can be determined by using the boundary condition that the liquid velocity at $y = 0$ equals that of the moving photographic film. That is, $v_x = U$ at $y = 0$, yielding $c_2 = U$; thus, the final velocity profile is:

$$v_x = U - \alpha y \left(\delta - \frac{y}{2} \right). \tag{E6.3.13}$$

Observe that the velocity profile, which is parabolic, consists of two parts:

1. A constant and positive part, arising from the film velocity, U.
2. A variable and negative part, which reduces v_x at increasing distances y from the film and eventually causes it to become negative.

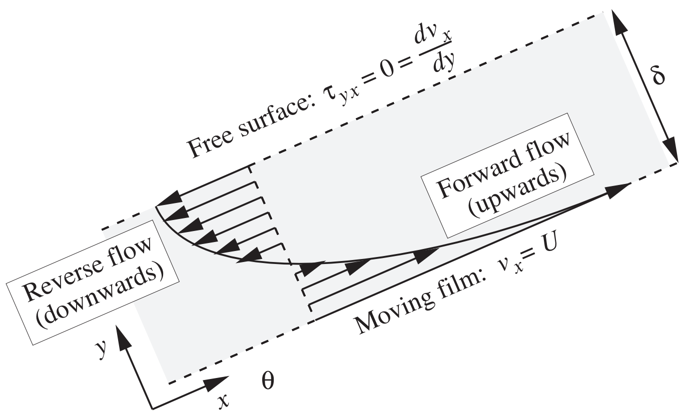


Fig. E6.3.2 Velocity profile in thin liquid layer on moving photographic film for the case of zero net liquid flow rate.

Exactly how much of the liquid is flowing upwards, and how much downwards, depends on the values of the variables U, δ, and α. However, we are asked to investigate the situation in which there is no net flow of liquid—that is, as much is being pulled up by the film as is falling back by gravity. In this case:

$$Q = \int_0^\delta v_x \, dy = \int_0^\delta \left[U - \alpha y \left(\delta - \frac{y}{2} \right) \right] dy = U\delta - \frac{1}{3}\alpha\delta^3 = 0, \tag{E6.3.14}$$

giving the thickness of the liquid film as:

$$\delta = \sqrt{\frac{3U}{\alpha}}. \tag{E6.3.15}$$

The velocity profile for this case of $Q = 0$ is shown in Fig. E6.3.2. □

6.4 Poiseuille and Couette Flows in Polymer Processing

The study of *polymer processing* falls into the realm of the chemical engineer. First, the polymer, such as nylon, polystyrene, or polyethylene, is produced by a chemical reaction—either as a liquid or solid. (In the latter event, it would subsequently have to be melted in order to be processed further.) Second, the polymer must be formed by suitable equipment into the desired final shape, such as a film, fiber, bottle, or other molded object. The procedures listed in Table 6.1 are typical of those occurring in polymer processing.

Table 6.1 Typical Polymer-Processing Operations

Operation	*Description*
Extrusion and die flow	The polymer is forced, either by an applied pressure, or pump, or a rotating screw, through a narrow opening—called a *die*—in order to form a continuous sheet, filament, or tube.
Drawing or "Spinning"	The polymer flows out through a narrow opening, either as a sheet or a thread, and is pulled by a roller in order to make a thinner sheet or thread.
Injection molding	The polymer is forced under high pressure into a mold, in order to form a variety of objects, such as telephones and automobile bumpers.
Blow molding	A "balloon" of polymer is expanded by the pressure of a gas in order to fill a mold. Bottles are typically formed by blow molding.
Calendering	The polymer is forced through two rotating rollers in order to form a relatively thin sheet. The nature of the surfaces of the rollers will strongly influence the final appearance of the sheet, which may be smooth, rough, or have a pattern embossed on it.
Coating	The polymer is applied as a thin film by a blade or rollers on to a substrate, such as paper or a sheet of another polymer.

Since polymers are generally highly viscous, their flows can be obtained by solving the equations of motion. In this chapter, we cover the rudiments of extrusion, die flow, and drawing or spinning. The analysis of calendering and coating is considerably more complicated, but can be rendered tractable if reasonable simplifications, known collectively as the *lubrication approximation,* are made, as discussed in Chapter 8.

Example 6.4—The Screw Extruder

Because polymers are generally highly viscous, they often need very high pressures to push them through dies. One such "pump" for achieving this is the *screw* *extruder,* shown in Fig. E6.4.1. The polymer typically enters the feed hopper as pellets, and is pushed forward by the screw, which rotates at an angular velocity ω, clockwise as seen by an observer looking along the axis from the inlet to the exit. The heated barrel melts the pellets, which then become fluid as the *metering section* of length L_0 is encountered (where the screw radius is r and the gap between the screw and barrel is h, with $h \ll r$). The screw increases the pressure of the polymer melt, which ultimately passes to a die at the exit of the extruder. The preliminary analysis given here neglects any heat-transfer effects in the metering section, and also assumes that the polymer has a constant Newtonian viscosity μ.

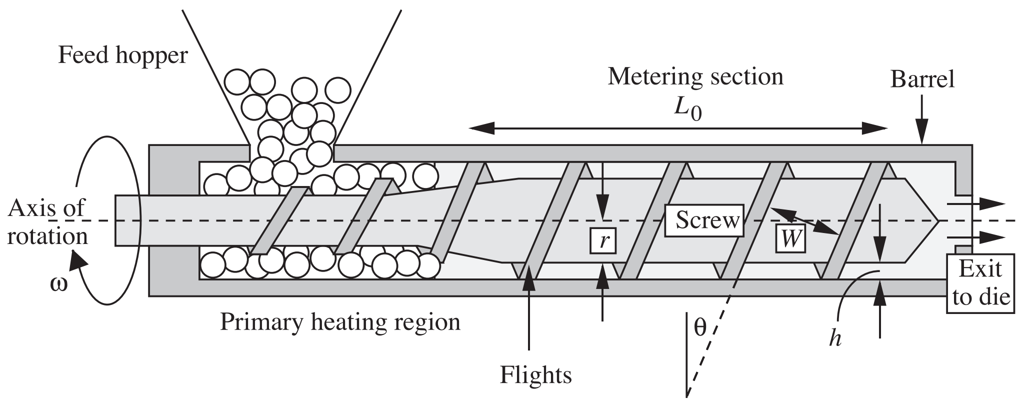


Fig. E6.4.1 Screw extruder.

The investigation is facilitated by taking the viewpoint of a hypothetical observer *located on the screw,* in which case the screw surface and the flights appear to be stationary, with the barrel moving with velocity $V = r\omega$ at an angle θ to the flight axis, as shown in Fig. E6.4.2. The alternative viewpoint of an observer on the inside surface of the barrel is not very fruitful, because not only are the flights seen as *moving* boundaries, but the observations would be periodically *blocked* as the flights passed over the observer!

Solution

Motion in two principal directions is considered:

1. Flow *parallel* to the flight axis, caused by a barrel velocity of $V_y = V\cos\theta = r\omega\cos\theta$ relative to the (now effectively stationary) flights and screw.
2. Flow *normal* to the flight axis, caused by a barrel velocity of $V_x = -V\sin\theta = -r\omega\sin\theta$ relative to the (stationary) flights and screw.

In each case, the flow is considered one-dimensional, with "end-effects" caused by the presence of the flights being unimportant. A glance at Fig. E6.4.3(b) will give the general idea. Although the flow in the x-direction must reverse itself as it nears the flights, it is reasonable to assume for $h \ll W$ that there is a substantial central region in which the flow is essentially in the positive or negative x-direction.

1. Motion parallel to the flight axis. The reader may wish to investigate the additional simplifying assumptions that give the y-momentum balance as:

$$\frac{\partial p}{\partial y} = \mu \frac{d^2 v_y}{dz^2}. \tag{E6.4.1}$$

Integration twice yields the velocity profile as:

$$v_y = \frac{1}{2\mu}\left(\frac{\partial p}{\partial y}\right) z^2 + c_1 z + c_2 = \underbrace{\frac{1}{2\mu}\left(-\frac{\partial p}{\partial y}\right)(hz - z^2)}_{\text{Poiseuille flow}} + \underbrace{\frac{z}{h} r\omega \cos\theta}_{\text{Couette flow}}. \tag{E6.4.2}$$

Here, the integration constants c_1 and c_2 have been determined in the usual way by applying the boundary conditions:

$$z = 0: \quad v_y = 0; \qquad z = h: \quad v_y = V_y = V\cos\theta = r\omega\cos\theta. \tag{E6.4.3}$$

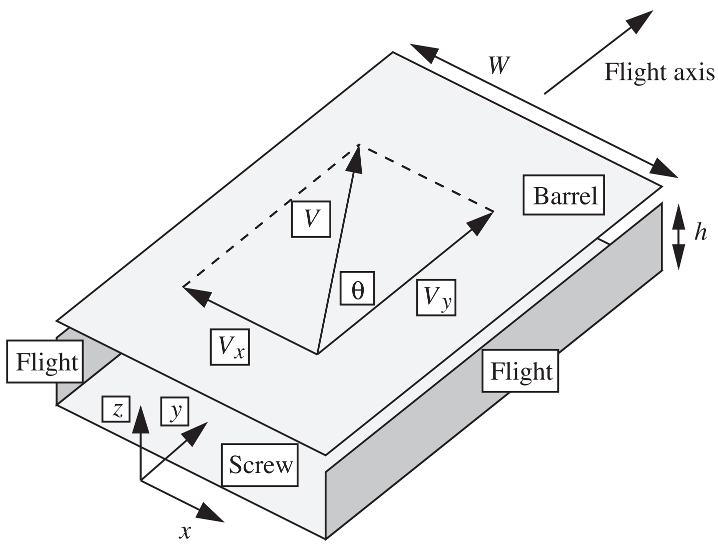
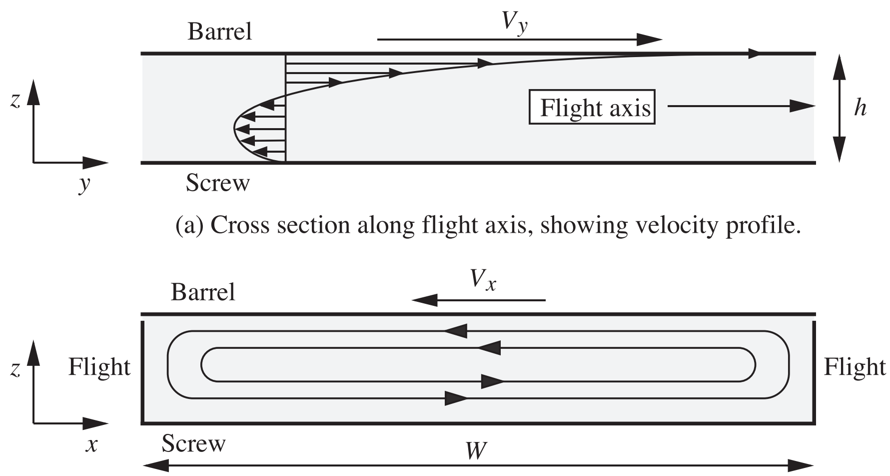


Fig. E6.4.2 Diagonal motion of barrel relative to flights.

Note that the negative of the pressure gradient is given in terms of the inlet pressure p_1, the exit pressure p_2, and the total length L $(= L_0/\sin\theta)$ measured along the screw flight axis by:

$$-\frac{\partial p}{\partial y} = -\frac{p_2 - p_1}{L} = \frac{p_1 - p_2}{L}, \tag{E6.4.4}$$

and is a negative quantity since the screw action builds up pressure and $p_2 > p_1$. Thus, Eqn. (E6.4.2) predicts a Poiseuille-type *backflow* (caused by the adverse pressure gradient) and a Couette-type *forward flow* (caused by the relative motion of the barrel to the screw). The combination is shown in Fig. E6.4.3(a).



(a) Cross section along flight axis, showing velocity profile.

(b) Cross section normal to flight axis, showing streamlines.

Fig. E6.4.3 Fluid motion (a) along and (b) normal to the flight axis, as seen by an observer on the screw.

The total flow rate Q_y of polymer melt in the direction of the flight axis is obtained by integrating the velocity between the screw and barrel, and recognizing that the width between flights is W:

$$Q_y = W \int_0^h v_y \, dz = \underbrace{\frac{Wh^3}{12\mu}\left(\frac{p_1 - p_2}{L}\right)}_{\text{Poiseuille}} + \underbrace{\frac{1}{2}\, Whr\omega \cos\theta}_{\text{Couette}}. \tag{E6.4.5}$$

The actual value of Q_y will depend on the resistance of the die located at the extruder exit. In a hypothetical case, in which the die offers no resistance, there would be no pressure increase in the extruder ($p_2 = p_1$), leaving only the Couette term in Eqn. (E6.4.5). For the practical situation in which the die offers significant resistance, the Poiseuille term would serve to *diminish* the flow rate given by the Couette term.

2. Motion normal to the flight axis. By a development very similar to that for flow parallel to the flight axis, we obtain:

$$\frac{\partial p}{\partial x} = \mu \frac{d^2 v_x}{dz^2}, \tag{E6.4.6}$$

$$v_x = \underbrace{\frac{1}{2\mu}\left(-\frac{\partial p}{\partial x}\right)(hz - z^2)}_{\text{Poiseuille flow}} - \underbrace{\frac{z}{h} r\omega \sin\theta}_{\text{Couette flow}}, \tag{E6.4.7}$$

$$Q_x = \int_0^h v_x \, dz = \underbrace{\frac{h^3}{12\mu}\left(-\frac{\partial p}{\partial x}\right)}_{\text{Poiseuille}} - \underbrace{\frac{1}{2}\, hr\omega \sin\theta}_{\text{Couette}} = 0. \tag{E6.4.8}$$

Here, Q_x is the flow rate in the x-direction, per unit depth along the flight axis, and must equal zero, because the flights at either end of the path act as barriers. The negative of the pressure gradient is therefore:

$$-\frac{\partial p}{\partial x} = \frac{6\mu r\omega \sin\theta}{h^2}, \tag{E6.4.9}$$

so that the velocity profile is given by:

$$v_x = \frac{z}{h} r\omega \left(2 - 3\frac{z}{h}\right) \sin\theta. \tag{E6.4.10}$$

Note from Eqn. (E6.4.10) that v_x is zero when either $z = 0$ (on the screw surface) or $z/h = 2/3$. The reader may wish to sketch the general appearance of $v_z(z)$. □

6.5 Solution of the Equations of Motion in Cylindrical Coordinates

Several chemical engineering operations exhibit symmetry about an axis z and involve one or more surfaces that can be described by having a constant radius for a given value of z. Examples are flow in pipes, extrusion of fibers, and viscometers that involve flow between concentric cylinders, one of which is rotating. Such cases lend themselves naturally to solution in *cylindrical* coordinates, and two examples will now be given.

Example 6.5—Flow Through an Annular Die

Following the discussion of polymer processing in the previous section, now consider flow through a die that could be located at the exit of the screw extruder of Example 6.4. Consider a die that forms a *tube* of polymer (other shapes being sheets and filaments). In the die of length D shown in Fig. E6.5, a pressure difference $p_2 - p_3$ causes a liquid of viscosity μ to flow steadily from left to right in the annular area between two fixed concentric cylinders. Note that p_2 is chosen for the inlet pressure in order to correspond to the extruder exit pressure from Example 6.4. The inner cylinder is solid, whereas the outer one is hollow; their radii are r_1 and r_2, respectively. The problem, which could occur in the extrusion of plastic tubes, is to find the velocity profile in the annular space and the total volumetric flow rate Q. Note that *cylindrical* coordinates are now involved.

Assumptions and continuity equation. The following assumptions are realistic:

1. There is only one nonzero velocity component, namely that in the direction of flow, v_z. Thus, $v_r = v_\theta = 0$.
2. Gravity acts vertically downwards, so that $g_z = 0$.
3. The axial velocity is independent of the angular location; that is, $\partial v_z/\partial\theta = 0$.

To analyze the situation, again start from the continuity equation, (5.49):

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial(\rho r v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} = 0, \tag{5.49}$$

which, for constant density and $v_r = v_\theta = 0$, reduces to:

$$\frac{\partial v_z}{\partial z} = 0, \tag{E6.5.1}$$

verifying that v_z is independent of distance from the inlet, and that the velocity profile $v_z = v_z(r)$ appears the same for all values of z.

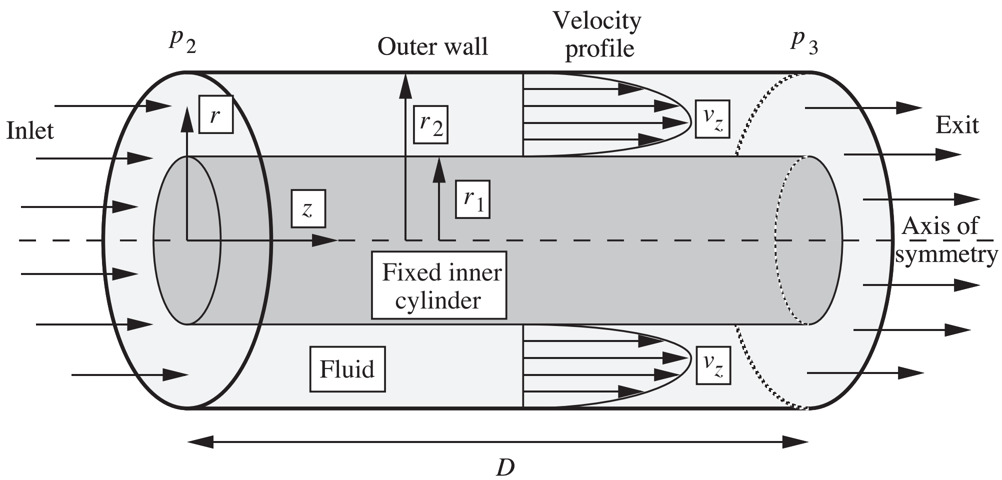


Fig. E6.5 Geometry for flow through an annular die.

Momentum balances. There are again three momentum balances, one for each of the r, θ, and z directions. If explored, the first two of these would ultimately lead to the pressure variation with r and θ at any cross section, which is of little interest in this problem. Therefore, we extract from Eqn. (5.75) only the z momentum balance:

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right)$$
$$= -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z. \tag{E6.5.2}$$

With $v_r = v_\theta = 0$ (from assumption 1), $\partial v_z/\partial z = 0$ [from Eqn. (E6.5.1)], $\partial v_z/\partial \theta = 0$ (assumption 3), and $g_z = 0$ (assumption 2), this momentum balance simplifies to:

$$\mu\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)\right] = \frac{\partial p}{\partial z}, \tag{E6.5.3}$$

in which total derivatives are used because v_z depends only on r.

Shortly, we shall prove that the pressure gradient is uniform between the die inlet and exit, being given by:

$$-\frac{\partial p}{\partial z} = \frac{p_2 - p_3}{D}, \tag{E6.5.4}$$

in which both sides of the equation are positive quantities. Two successive integrations of Eqn. (E6.5.3) may then be performed, yielding:

$$v_z = -\frac{1}{4\mu}\left(-\frac{\partial p}{\partial z}\right) r^2 + c_1 \ln r + c_2. \tag{E6.5.5}$$

The two constants may be evaluated by applying the boundary conditions of zero velocity at the inner and outer walls,

$$r = r_1\colon\ v_z = 0, \qquad r = r_2\colon\ v_z = 0, \tag{E6.5.6}$$

giving:

$$c_1 = \frac{1}{4\mu}\left(-\frac{\partial p}{\partial z}\right)\frac{r_2^2 - r_1^2}{\ln(r_2/r_1)}, \qquad c_2 = \frac{1}{4\mu}\left(-\frac{\partial p}{\partial z}\right) r_2^2 - c_1 \ln r_2. \tag{E6.5.7}$$

Substitution of these values for the constants of integration into Eqn. (E6.5.5) yields the final expression for the velocity profile:

$$v_z = \frac{1}{4\mu}\left(-\frac{\partial p}{\partial z}\right)\left[\frac{\ln(r/r_1)}{\ln(r_2/r_1)}(r_2^2 - r_1^2) - (r^2 - r_1^2)\right], \tag{E6.5.8}$$

which is sketched in Fig. E6.5. Note that the maximum velocity occurs somewhat *before* the halfway point in progressing from the inner cylinder to the outer cylinder.

Volumetric flow rate. The final quantity of interest is the volumetric flow rate Q. Observing first that the flow rate through an annulus of internal radius r and external radius $r + dr$ is $dQ = v_z 2\pi r\, dr$, integration yields:

$$Q = \int_0^Q dQ = \int_{r_1}^{r_2} v_z 2\pi r\, dr. \tag{E6.5.9}$$

Since $r \ln r$ is involved in the expression for v_z, the following indefinite integral is needed:

$$\int r \ln r\, dr = \frac{r^2}{2}\ln r - \frac{r^2}{4}, \tag{E6.5.10}$$

giving the final result:

$$Q = \frac{\pi(r_2^2 - r_1^2)}{8\mu}\left(-\frac{\partial p}{\partial z}\right)\left[r_2^2 + r_1^2 - \frac{r_2^2 - r_1^2}{\ln(r_2/r_1)}\right]. \tag{E6.5.11}$$

Since Q, μ, r_1, and r_2 are constant throughout the die, $\partial p/\partial z$ is also constant, thus verifying the hypothesis previously made. Observe that in the limiting case of $r_1 \to 0$, Eqn. (E6.5.11) simplifies to the Hagen-Poiseuille law, already stated in Eqn. (3.12).

This problem may also be solved by performing a momentum balance on a shell that consists of an *annulus* of internal radius r, external radius $r + dr$, and length dz. □

Example 6.6—Spinning a Polymeric Fiber

A Newtonian polymeric liquid of viscosity μ is being "spun" (drawn into a fiber or filament of small diameter before solidifying by pulling it through a chemical setting bath) in the apparatus shown in Fig. E6.6.

The liquid volumetric flow rate is Q, and the filament diameters at $z = 0$ and $z = L$ are D_0 and D_L, respectively. To a first approximation, the effects of gravity, inertia, and surface tension are negligible. Derive an expression for the tensile force F needed to pull the filament downwards. Assume that the axial velocity profile is "flat" at any vertical location, so that v_z depends only on z, which is here most conveniently taken as positive in the *downwards* direction. Also derive an expression for the downwards velocity v_z as a function of z. The inset of Fig. E6.6 shows further details of the notation concerning the filament.

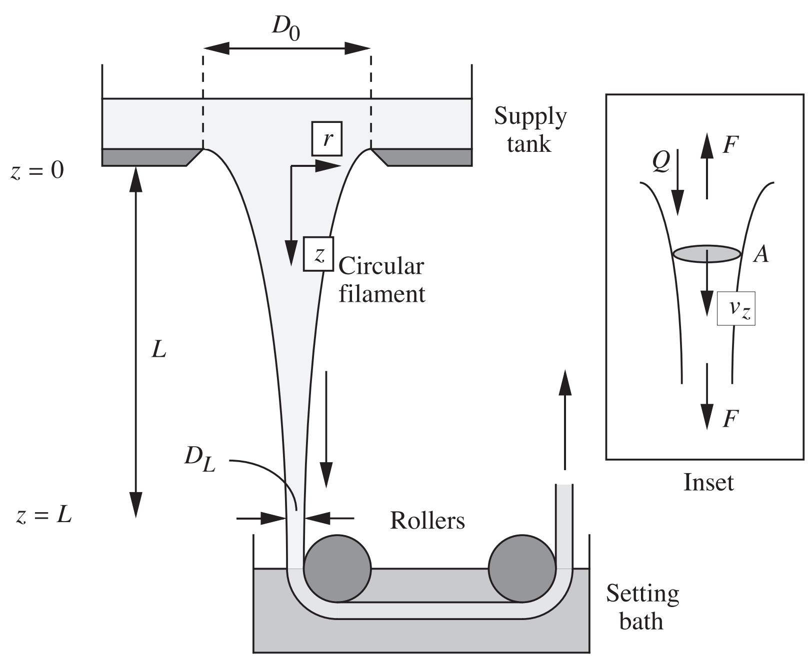


Fig. E6.6 "Spinning" a polymer filament, whose diameter in relation to its length is exaggerated in the diagram.

Solution

It is first necessary to determine the radial velocity and hence the pressure inside the filament. From continuity:

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r} + \frac{\partial v_z}{\partial z} = 0. \tag{E6.6.1}$$

Since v_z depends only on z, its axial derivative is a function of z only, or $dv_z/dz = f(z)$, so that Eqn. (E6.6.1) may be rearranged and integrated at constant z to give:

$$\frac{\partial(rv_r)}{\partial r} = -rf(z), \qquad rv_r = -\frac{r^2 f(z)}{2} + g(z). \tag{E6.6.2}$$

But to avoid an infinite value of v_r at the centerline, g(z) must be zero, giving:

$$v_r = -\frac{rf(z)}{2}, \qquad \frac{\partial v_r}{\partial r} = -\frac{f(z)}{2}. \tag{E6.6.3}$$

To proceed with reasonable expediency, it is necessary to make some simplification. After accounting for a primary effect (the difference between the pressure in the filament and the surrounding atmosphere), we assume that a secondary effect (variation of pressure across the filament) is negligible; that is, the pressure does not depend on the radial location. Noting that the external (gauge) pressure is zero everywhere, and applying the first part of Eqn. (5.64) at the free surface:

$$\sigma_{rr} = -p + 2\mu\frac{\partial v_r}{\partial r} = -p - \mu f(z) = 0, \quad \text{or} \quad p = -\mu\frac{dv_z}{dz}. \tag{E6.6.4}$$

The axial stress is therefore:

$$\sigma_{zz} = -p + 2\mu\frac{dv_z}{dz} = 3\mu\frac{dv_z}{dz}. \tag{E6.6.5}$$

It is interesting to note that the same result can be obtained with an alternative assumption.[2] The axial tension in the fiber equals the product of the cross-sectional area and the local axial stress:

$$F = A\sigma_{zz} = 3\mu A\frac{dv_z}{dz}. \tag{E6.6.6}$$

Since the effect of gravity is stated to be insignificant, F is a constant, regardless of the vertical location.

At any location, the volumetric flow rate equals the product of the cross-sectional area and the axial velocity:

$$Q = Av_z. \tag{E6.6.7}$$

[2] See page 235 of S. Middleman's *Fundamentals of Polymer Processing*, McGraw-Hill, New York, 1977. There, the author assumes $\sigma_{rr} = \sigma_{\theta\theta} = 0$, followed by the identity: $p = -(\sigma_{zz} + \sigma_{rr} + \sigma_{\theta\theta})/3$.

A differential equation for the velocity is next obtained by dividing one of the last two equations by the other, and rearranging:

$$\frac{1}{v_z}\frac{dv_z}{dz} = \frac{F}{3\mu Q}. \tag{E6.6.8}$$

Integration, noting that the inlet velocity at $z = 0$ is $v_{z0} = Q/(\pi D_0^2/4)$, gives:

$$\int_{v_{z0}}^{v_z} \frac{dv_z}{v_z} = \frac{F}{3\mu Q}\int_0^z dz, \qquad \text{where} \quad v_{z0} = \frac{4Q}{\pi D_0^2}, \tag{E6.6.9}$$

so that the axial velocity obeys:

$$v_z = v_{z0} e^{Fz/3\mu Q}. \tag{E6.6.10}$$

The tension is obtained by applying Eqns. (E6.6.7) and (E6.6.10) just before the filament is taken up by the rollers:

$$v_{zL} = \frac{4Q}{\pi D_L^2} = v_{z0} e^{FL/3\mu Q}. \tag{E6.6.11}$$

Rearrangement yields:

$$F = \frac{3\mu Q}{L} \ln \frac{v_{zL}}{v_{z0}}, \tag{E6.6.12}$$

which predicts a force that *increases* with higher viscosities, flow rates, and draw-down ratios (v_{zL}/v_{z0}), and that *decreases* with longer filaments.

Elimination of F from Eqns. (E6.6.10) and (E6.6.12) gives an expression for the velocity that depends only on the variables specified originally:

$$v_z = v_{z0}\left(\frac{v_{zL}}{v_{z0}}\right)^{z/L} = v_{z0}\left(\frac{D_0}{D_L}\right)^{2z/L}, \tag{E6.6.13}$$

a result that is independent of the viscosity. □

6.6 Solution of the Equations of Motion in Spherical Coordinates

Most of the introductory viscous-flow problems will lend themselves to solution in either rectangular or cylindrical coordinates. Occasionally, as in Example 6.7, a problem will arise in which *spherical* coordinates should be used. It is a fairly advanced problem! Try first to appreciate the broad steps involved, and then peruse the fine detail at a second reading.

Example 6.7—Analysis of a Cone-and-Plate Rheometer

The problem concerns the analysis of a cone-and-plate rheometer, an instrument developed and perfected in the 1950s and 1960s by Prof. Karl Weissenberg, for measuring the viscosity of liquids, and also known as the "Weissenberg rheogoniometer."[3] A cross section of the essential features is shown in Fig. E6.7, in which the liquid sample is held by surface tension in the narrow opening between a rotating lower circular plate, of radius R, and an upper cone, making an angle of β with the vertical axis. The plate is rotated steadily in the ϕ direction with an angular velocity ω, causing the liquid in the gap to move in concentric circles with a velocity v_ϕ. (In practice, the tip of the cone is slightly truncated, to avoid friction with the plate.) Observe that the flow is of the *Couette* type.

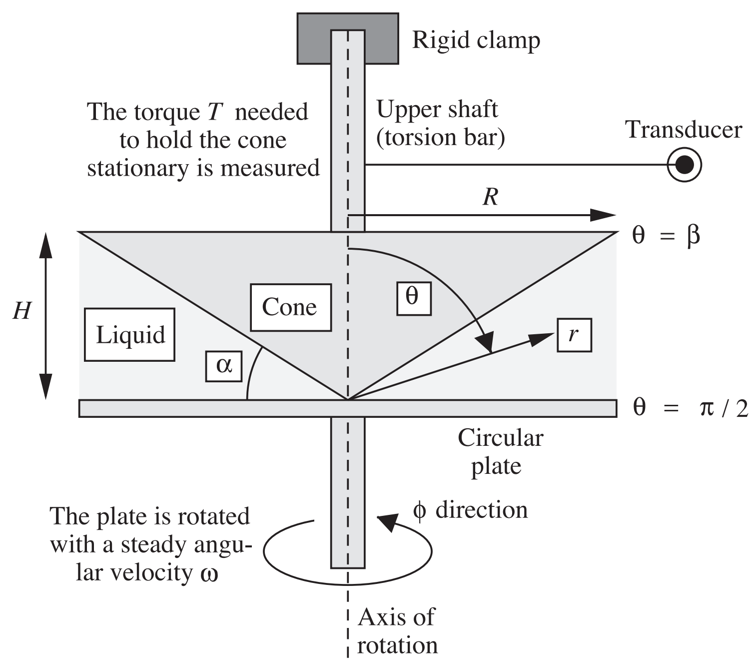


Fig. E6.7 Geometry for a Weissenberg rheogoniometer. (The angle between the cone and plate is exaggerated.)

The top of the upper shaft—which acts like a torsion bar—is clamped rigidly. However, viscous friction will twist the cone and the lower portions of the upper shaft very slightly; the amount of motion can be detected by a light arm at the extremity of which is a transducer, consisting of a small piece of steel, attached to the arm, and surrounded by a coil of wire; by monitoring the inductance of the coil,

[3] Professor Weissenberg once related to the author that he (Prof. W.) was attending an instrument trade show in London. There, the rheogoniometer was being demonstrated by a young salesman who was unaware of Prof. W's identity. Upon inquiring how the instrument worked, the salesman replied: "I'm sorry, sir, but it's quite complicated, and I don't think you will be able to understand it."

the small angle of twist can be obtained; a knowledge of the elastic properties of the shaft then enables the restraining torque T to be obtained. From the analysis given below, it is then possible to deduce the viscosity of the sample. The instrument is so sensitive that if no liquid is present, it is capable of determining the viscosity of the air in the gap!

The problem is best solved using *spherical* coordinates, because the surfaces of the cone and plate are then described by *constant* values of the angle θ, namely β and $\pi/2$, respectively.

Assumptions and the continuity equation. The following realistic assumptions are made:

1. There is only one nonzero velocity component, namely that in the ϕ direction, v_ϕ. Thus, $v_r = v_\theta = 0$.
2. Gravity acts vertically downwards, so that $g_\phi = 0$.
3. We do not need to know how the pressure varies in the liquid. Therefore, the r and θ momentum balances, which would supply this information, are not required.

The analysis starts once more from the continuity equation, (5.50):

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial(\rho r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\rho v_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial(\rho v_\phi)}{\partial \phi} = 0, \tag{5.50}$$

which, for constant density and $v_r = v_\theta = 0$ reduces to:

$$\frac{\partial v_\phi}{\partial \phi} = 0, \tag{E6.7.1}$$

verifying that v_ϕ is independent of the angular location ϕ, so we are correct in examining just one representative cross section, as shown in Fig. E6.7.

Momentum balances. There are again three momentum balances, one for each of the r, θ, and ϕ directions. From the third assumption above, the first two such balances are of no significant interest, leaving, from Eqn. (5.77), just that in the ϕ direction:

$$\begin{aligned}
&\rho\left(\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r}{r} + \frac{v_\theta v_\phi \cot\theta}{r}\right)\\
&\quad = -\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi} + \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v_\phi}{\partial \theta}\right)\right. \qquad \text{(E6.7.2)}\\
&\qquad \left. + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial \phi^2} - \frac{v_\phi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial \phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\theta}{\partial \phi}\right] + \rho g_\phi.
\end{aligned}$$

With $v_r = v_\theta = 0$ (assumption 1), $\partial v_\phi/\partial\phi = 0$ [from Eqn. (E6.7.1)], and $g_\phi = 0$ (assumption 2), the momentum balance simplifies to:

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v_\phi}{\partial \theta}\right) - \frac{v_\phi}{\sin^2\theta} = 0. \tag{E6.7.3}$$

Determination of the velocity profile. First, seek an expression for the velocity in the ϕ direction, which is expected to be proportional to both the distance r from the origin and the angular velocity ω of the lower plate. However, its variation with the coordinate θ is something that has to be discovered. Therefore, postulate a solution of the form:

$$v_\phi = r\omega f(\theta), \tag{E6.7.4}$$

in which the function $f(\theta)$ is to be determined. Substitution of v_ϕ from Eqn. (E6.7.4) into Eqn. (E6.7.3), and using $f'$and f'' to denote the first and second *total* derivatives of f with respect to θ, gives:

$$\begin{aligned} & f'' + f' \cot\theta + f(1 - \cot^2\theta) \\ & \quad \equiv \frac{1}{\sin^2\theta}\frac{d}{d\theta}(f' \sin^2\theta - f\sin\theta\cos\theta) \\ & \quad = \frac{1}{\sin^2\theta}\frac{d}{d\theta}\left[\sin^3\theta\frac{d}{d\theta}\left(\frac{f}{\sin\theta}\right)\right] = 0. \end{aligned} \tag{E6.7.5}$$

The reader is encouraged, as always, to check the missing algebraic and trigonometric steps, although they are rather tricky here![4]

By multiplying Eqn. (E6.7.5) through by $\sin^2\theta$, it follows after integration that the quantity in brackets is a constant, here represented as $-2c_1$, the reason for the "-2" being that it will cancel with a similar factor later on:

$$\sin^3\theta\frac{d}{d\theta}\left(\frac{f}{\sin\theta}\right) = -2c_1. \tag{E6.7.6}$$

Separation of variables and indefinite integration (without specified limits) yields:

$$\int d\left(\frac{f}{\sin\theta}\right) = \frac{f}{\sin\theta} = -2c_1\int\frac{d\theta}{\sin^3\theta}. \tag{E6.7.7}$$

To proceed further, we need the following two standard indefinite integrals and one trigonometric identity:[5]

$$\int\frac{d\theta}{\sin^3\theta} = -\frac{1}{2}\frac{\cot\theta}{\sin\theta} + \frac{1}{2}\int\frac{d\theta}{\sin\theta}, \tag{E6.7.8}$$

$$\int\frac{d\theta}{\sin\theta} = \ln\left(\tan\frac{\theta}{2}\right) = \frac{1}{2}\ln\left(\tan^2\frac{\theta}{2}\right), \tag{E6.7.9}$$

$$\tan^2\frac{\theta}{2} = \frac{1-\cos\theta}{1+\cos\theta}. \tag{E6.7.10}$$

[4] Although our approach is significantly different from that given on page 98 *et seq.* of Bird, R.B., Stewart, W.E., and Lightfoot, E.N., *Transport Phenomena*, Wiley, New York (1960), we are indebted to these authors for the helpful hint they gave in establishing the equivalency expressed in our Eqn. (E6.7.5).

[5] From pages 87 (integrals) and 72 (trigonometric identity) of Perry, J.H., ed., *Chemical Engineers' Handbook*, 3rd ed., McGraw-Hill, New York (1950).

Armed with these, Eqn. (E6.7.7) leads to the following expression for f:

$$f = c_1 \left[\cot\theta + \frac{1}{2} \left(\ln \frac{1+\cos\theta}{1-\cos\theta} \right) \sin\theta \right] + c_2, \tag{E6.7.11}$$

in which c_2 is a second constant of integration.

Implementation of the boundary conditions. The constants c_1 and c_2 are found by imposing the two boundary conditions:

1. At the lower plate, where $\theta = \pi/2$ and the expression in parentheses in Eqn. (E6.7.11) is zero, so that $f = c_2$, the velocity is simply the radius times the angular velocity:

$$v_\phi \equiv r\omega f = r\omega c_2 = r\omega, \quad \text{or} \quad c_2 = 1. \tag{E6.7.12}$$

2. At the surface of the cone, where $\theta = \beta$, the velocity $v_\phi = r\omega f$ is zero. Hence $f = 0$, and Eqn. (E6.7.11) leads to:

$$c_1 = -c_2 g(\beta) = -g(\beta), \ \text{ where } \ \frac{1}{g(\beta)} = \cot\beta + \frac{1}{2} \left(\ln \frac{1+\cos\beta}{1-\cos\beta} \right) \sin\beta. \tag{E6.7.13}$$

Substitution of these expressions for c_1 and c_2 into Eqn. (E6.7.11), and noting that $v_\phi = r\omega f$, gives the final (!) expression for the velocity:

$$v_\phi = r\omega \left[1 - \frac{\cot\theta + \dfrac{1}{2} \left(\ln \dfrac{1+\cos\theta}{1-\cos\theta} \right) \sin\theta}{\cot\beta + \dfrac{1}{2} \left(\ln \dfrac{1+\cos\beta}{1-\cos\beta} \right) \sin\beta} \right]. \tag{E6.7.14}$$

As a partial check on the result, note that Eqn. (E6.7.14) reduces to $v_\phi = r\omega$ when $\theta = \pi/2$ and to $v_\phi = 0$ when $\theta = \beta$.

Shear stress and torque. Recall that the primary goal of this investigation is to determine the torque T needed to hold the cone stationary. The relevant shear stress exerted by the liquid on the surface of the cone is $\tau_{\theta\phi}$—that exerted on the under surface of the cone (of constant first subscript, $\theta = \beta$) in the positive ϕ direction (refer again to Fig. 5.13 for the sign convention and notation for stresses). Since this direction is the same as that of the rotation of the lower plate, we expect that $\tau_{\theta\phi}$ will prove to be positive, thus indicating that the liquid is trying to turn the cone in the same direction in which the lower plate is rotated.

From the second of Eqn. (5.65), the relation for this shear stress is:

$$\tau_{\theta\phi} = \mu \left[\frac{\sin\theta}{r} \frac{\partial}{\partial\theta} \left(\frac{v_\phi}{\sin\theta} \right) + \frac{1}{r\sin\theta} \frac{\partial v_\theta}{\partial\phi} \right]. \tag{E6.7.15}$$

Since $v_\theta = 0$, and recalling Eqns. (E6.7.4), (E6.7.6), and (E6.7.13), the shear stress on the cone becomes:

$$(\tau_{\theta\phi})_{\theta=\beta} = \mu \left[\frac{\sin\theta}{r} \frac{\partial}{\partial\theta} \left(\frac{r\omega f}{\sin\theta} \right) \right]_{\theta=\beta} = -\left(\frac{2c_1\omega\mu}{\sin^2\theta} \right)_{\theta=\beta} = \frac{2\omega\mu g(\beta)}{\sin^2\beta}. \qquad \text{(E6.7.16)}$$

One importance of this result is that it is *independent* of r, giving a constant stress and strain throughout the liquid, a significant simplification when deciphering the experimental results for non-Newtonian fluids (see Chapter 11). In effect, the increased velocity differences between the plate and cone at the greater values of r are offset in exact proportion by the larger distances separating them.

The torque exerted by the liquid on the cone (in the positive ϕ direction) is obtained as follows. The surface area of the cone between radii r and $r + dr$ is $2\pi r \sin\beta\, dr$, and is located at a lever arm of $r \sin\beta$ from the axis of symmetry. Multiplication by the shear stress and integration gives:

$$T = \int_0^{R/\sin\beta} \underbrace{(2\pi r \sin\beta\, dr)}_{\text{Area}}\ \underbrace{r\sin\beta}_{\substack{\text{Lever}\\ \text{arm}}}\ \underbrace{(\tau_{\theta\phi})_{\theta=\beta}}_{\text{Stress}}, \qquad \text{(E6.7.17)}$$

Substitution of $(\tau_{\theta\phi})_{\theta=\beta}$ from Eqn. (E6.7.16) and integration gives the torque as:

$$T = \frac{4\pi\omega\mu g(\beta) R^3}{3\sin^3\beta}. \qquad \text{(E6.7.18)}$$

The torque for holding the cone stationary has the same value, but is, of course, in the *negative* ϕ direction.

Since R and $g(\beta)$ can be determined from the radius and the angle β of the cone in conjunction with Eqn. (E6.7.13), the viscosity μ of the liquid can finally be determined. □

Problems for Chapter 6

Unless otherwise stated, all flows are steady state, for a Newtonian fluid with constant density and viscosity.

1. *Stretching of a liquid film—M.* In broad terms, explain the meanings of the following two equations, paying attention to any sign convention:

$$\sigma_{xx} = -p + 2\mu \frac{\partial v_x}{\partial x}, \qquad \sigma_{yy} = -p + 2\mu \frac{\partial v_y}{\partial y}.$$

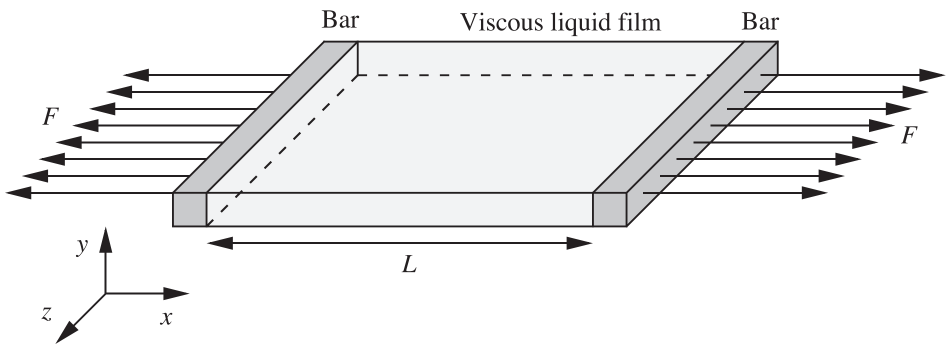


Fig. P6.1 Stretching of liquid film between two bars.

Fig. P6.1 shows a film of a viscous liquid held between two bars spaced a distance L apart. If the film thickness is uniform, and the total volume of liquid is V, show that the force necessary to separate the bars with a relative velocity dL/dt is:

$$F = \frac{4\mu V}{L^2}\frac{dL}{dt}.$$

2. *Wire coating—M.* Fig. P6.2 shows a rodlike wire of radius r_1 that is being pulled steadily with velocity V through a horizontal die of length L and internal radius r_2. The wire and the die are coaxial, and the space between them is filled with a liquid of viscosity μ. The pressure at both ends of the die is atmospheric. The wire is coated with the liquid as it leaves the die, and the thickness of the coating eventually settles down to a uniform value, δ.

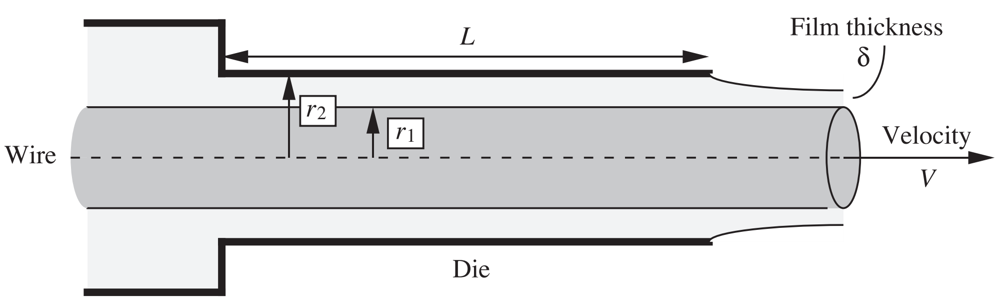


Fig. P6.2 Coating of wire drawn through a die.

Neglecting end effects, use the equations of motion in cylindrical coordinates to derive expressions for:

(a) The velocity profile within the annular space. Assume that there is only one nonzero velocity component, v_z, and that this depends only on radial position.
(b) The total volumetric flow rate Q through the annulus.
(c) The limiting value for Q if r_1 approaches zero.
(d) The final thickness, δ, of the coating on the wire.
(e) The force F needed to pull the wire.

3. *Off-center annular flow—D (C).* A liquid flows under a pressure gradient $\partial p/\partial z$ through the *narrow* annular space of a die, a cross section of which is shown in Fig. P6.3(a). The coordinate z is in the axial direction, normal to the plane of the diagram. The die consists of a solid inner cylinder with center P and radius b inside a hollow outer cylinder with center O and radius a. The points O and P were *intended* to coincide, but due to an imperfection of assembly are separated by a small distance δ.

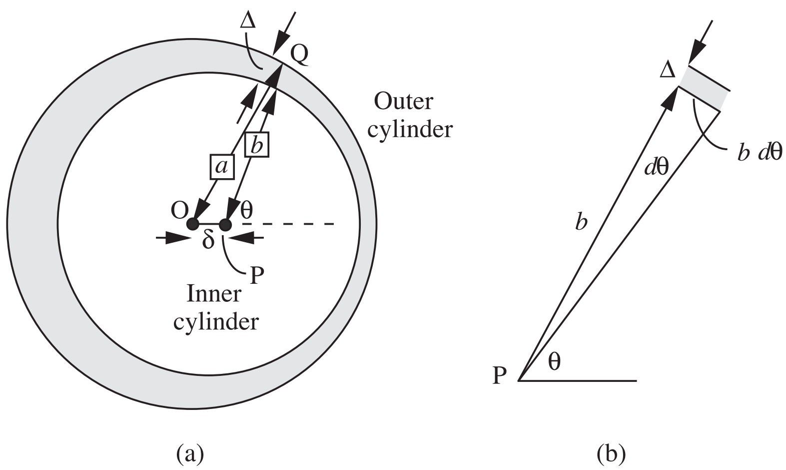


Fig. P6.3 Off-center cylinder inside a die (gap width exaggerated): (a) complete cross section; (b) effect of incrementing θ.

By a simple geometrical argument based on the triangle OPQ, show that the gap width Δ between the two cylinders is given approximately by:

$$\Delta \doteq a - b - \delta \cos\theta,$$

where the angle θ is defined in the diagram.

Now consider the radius arm b swung through an angle $d\theta$, so that it traces an arc of length $bd\theta$. The flow rate dQ through the shaded element in Fig. P6.3(b) is approximately that between parallel plates of width $bd\theta$ and separation Δ. Hence prove that the flow rate through the die is given approximately by:

$$Q = \pi bc(2\alpha^3 + 3\alpha\delta^2),$$

in which:

$$c = \frac{1}{12\mu}\left(-\frac{\partial p}{\partial z}\right), \qquad \text{and} \qquad \alpha = a - b.$$

Assume from Eqn. (E6.1.26) that the flow rate per unit width between two flat plates separated by a distance h is:

$$\frac{h^3}{12\mu}\left(-\frac{\partial p}{\partial z}\right).$$

What is the ratio of the flow rate if the two cylinders are touching at one point to the flow rate if they are concentric?

4. *Compression molding—M.* Fig. P6.4 shows the (a) beginning, (b) intermediate, and (c) final stages in the compression molding of a material that behaves as a liquid of high viscosity μ, from an initial cylinder of height H_0 and radius R_0 to a final disk of height H_1 and radius R_1.

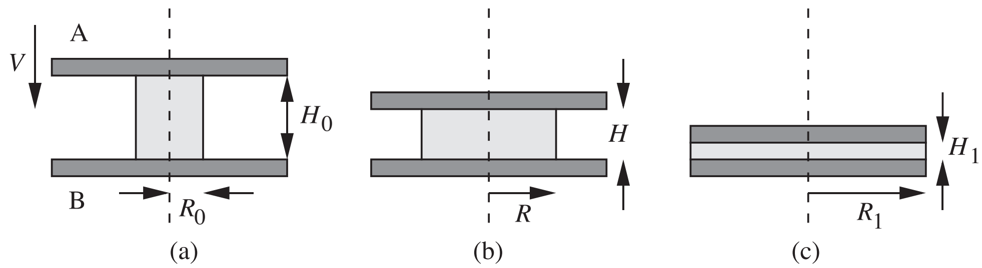


Fig. P6.4 Compression molding between two disks.

In the molding operation, the upper disk A is squeezed with a uniform velocity V towards the stationary lower disk B.

Ignoring small variations of pressure in the z direction, prove that the total compressive force F that must be exerted downwards on the upper disk is:

$$F = \frac{3\pi\mu V R^4}{2H^3}.$$

Assume that the liquid flow is radially outwards everywhere, with a parabolic velocity profile. Also assume from Eqn. (E6.1.26) that per unit width of a channel of depth H, the volumetric flow rate is:

$$Q = \frac{H^3}{12\mu}\left(-\frac{\partial p}{\partial r}\right).$$

Give expressions for H and R as functions of time t, and draw a sketch that shows how F varies with time.

5. *Film draining—M.* Fig. P6.5 shows an idealized view of a liquid film of viscosity μ that is draining under gravity down the side of a flat vertical wall. Such a situation would be *approximated* by the film left on the wall of a tank that was suddenly drained through a large hole in its base.

What are the justifications for assuming that the velocity profile at any distance x below the top of the wall is given by:

$$v_x = \frac{\rho g}{2\mu} y(2h - y),$$

where $h = h(x)$ is the local film thickness? Derive an expression for the corresponding downwards mass flow rate m per unit wall width (normal to the plane of the diagram).

Perform a *transient* mass balance on a differential element of the film and prove that h varies with time and position according to:

$$\frac{\partial h}{\partial t} = -\frac{1}{\rho}\frac{\partial m}{\partial x}.$$

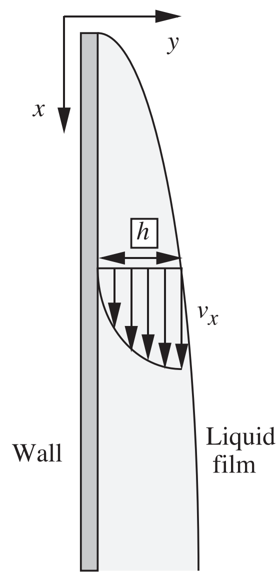


Fig. P6.5 Liquid draining from a vertical wall.

Now substitute your expression for m, to obtain a partial differential equation for h. Try a solution of the form:

$$h = c\,t^p x^q,$$

and determine the unknowns c, p, and q. Discuss the limitations of your solution.

Note that a similar situation occurs when a substrate is suddenly lifted from a bath of coating fluid.

6. *Sheet "spinning"—M.* A Newtonian polymeric liquid of viscosity μ is being "spun" (drawn into a sheet of small thickness before solidifying by pulling it through a chemical setting bath) in the apparatus shown in Fig. P6.6.

The liquid volumetric flow rate is Q, and the sheet thicknesses at $z = 0$ and $z = L$ are Δ and δ, respectively. The effects of gravity, inertia, and surface tension are negligible. Derive an expression for the tensile force needed to pull the filament downwards. *Hint*: start by assuming that the vertically downwards velocity v_z depends only on z and that the lateral velocity v_y is zero. Also derive an expression for the downwards velocity v_z as a function of z.

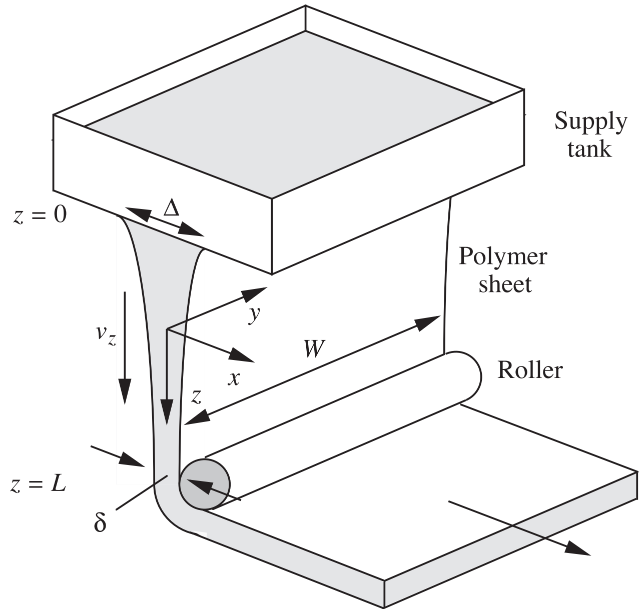


Fig. P6.6 "Spinning" a polymer sheet.

7. *Details of pipe flow—M.* A fluid of density ρ and viscosity μ flows from left to right through the horizontal pipe of radius a and length L shown in Fig. P6.7. The pressures at the centers of the inlet and exit are p_1 and p_2, respectively. You may assume that the only nonzero velocity component is v_z, and that this is not a function of the angular coordinate, θ.

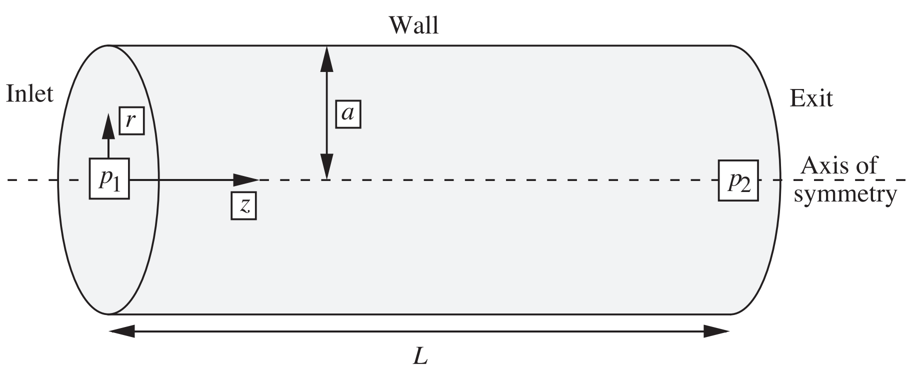


Fig. P6.7 Flow of a liquid in a horizontal pipe.

Stating any further necessary assumptions, derive expressions for the following, in terms of any or all of a, L, p_1, p_2, ρ, μ, and the coordinates r, z, and θ:

(a) The velocity profile, $v_z = v_z(r)$.
(b) The total volumetric flow rate Q through the pipe.

(c) The pressure p at any point (r, θ, z).
(d) The shear stress, τ_{rz}.

8. *Natural convection—M.* Fig. P6.8 shows two infinite parallel vertical walls that are separated by a distance $2d$. A fluid of viscosity μ and volume coefficient of expansion β fills the intervening space. The two walls are maintained at uniform temperatures T_1 (cold) and T_2 (hot), and you may assume (to be proved in a heat-transfer course) that there is a linear variation of temperature in the x direction. That is:

$$T = \overline{T} + \frac{x}{d}\left(\frac{T_2 - T_1}{2}\right), \qquad \text{where} \qquad \overline{T} = \frac{T_1 + T_2}{2}.$$

The density is *not* constant, but varies according to:

$$\rho = \overline{\rho}\left[1 - \beta\left(T - \overline{T}\right)\right],$$

where $\overline{\rho}$ is the density at the mean temperature $\overline{T}$, which occurs at $x = 0$.

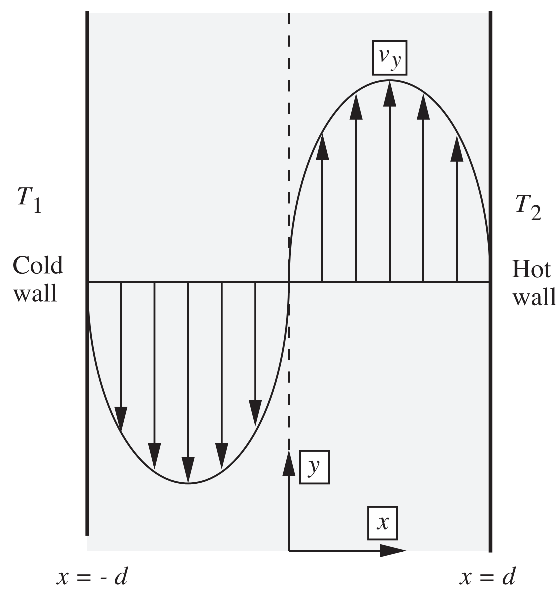


Fig. P6.8 Natural convection between vertical walls.

If the resulting natural-convection flow is steady, use the equations of motion to derive an expression for the velocity profile $v_y = v_y(x)$ between the plates. Your expression for v_y should be in terms of any or all of x, d, T_1, T_2, $\overline{\rho}$, μ, β, and g.

Hints: in the y momentum balance, you should find yourself facing the following combination:

$$-\frac{\partial p}{\partial y} + \rho g_y,$$

in which $g_y = -g$. These two terms are *almost* in balance, but not quite, leading to a small—but important—buoyancy effect that "drives" the natural convection.

The variation of pressure in the y direction may be taken as the normal hydrostatic variation:

$$\frac{\partial p}{\partial y} = -g\overline{\rho}.$$

We then have:

$$-\frac{\partial p}{\partial y} + \rho g_y = g\overline{\rho} + \overline{\rho}\left[1 - \beta\left(T - \overline{T}\right)\right](-g) = \beta g\left(T - \overline{T}\right),$$

and this will be found to be a vital contribution to the y momentum balance.

9. *Square duct velocity profile—M.* A certain flow in rectangular Cartesian coordinates has only one nonzero velocity component, v_z, and this does not vary with z. If there is no body force, write down the Navier-Stokes equation for the z momentum balance.

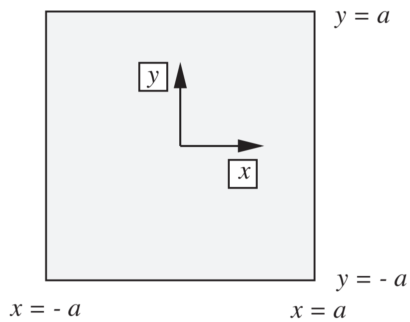


Fig. P6.9 Square cross section of a duct.

One-dimensional, fully developed steady flow occurs under a pressure gradient $\partial p/\partial z$ in the z direction, parallel to the axis of a square duct of side $2a$, whose cross section is shown in Fig. P6.9. The following equation has been proposed for the velocity profile:

$$v_z = \frac{1}{2\mu}\left(-\frac{\partial p}{\partial z}\right)a^2\left[1 - \left(\frac{x}{a}\right)^2\right]\left[1 - \left(\frac{y}{a}\right)^2\right].$$

Without attempting to integrate the momentum balance, investigate the possible merits of this proposed solution for v_z. Explain whether or not it is correct.

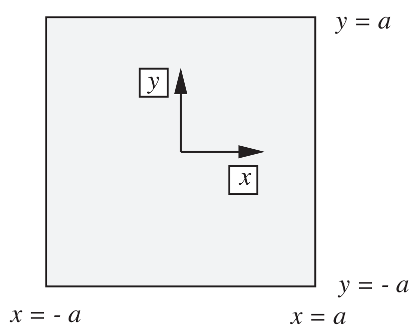


Fig. P6.10 Square cross section of a duct.

10. *Poisson's equation for a square duct—E.* A polymeric fluid of uniform viscosity μ is to be extruded after pumping it through a long duct whose cross section is a square of side $2a$, shown in Fig. P6.10. The flow is parallel everywhere to the axis of the duct, which is in the z direction, normal to the plane of the diagram.

If $\partial p/\partial z$, μ, and a are specified, show that the problem of obtaining the axial velocity distribution $v_z = v_z(x, y)$, amounts to solving Poisson's equation—of the form $\nabla^2 \phi = f(x, y)$, where f is specified and ϕ is the unknown. Also note that Poisson's equation can be solved numerically by the Matlab PDE (partial differential equation) Toolbox, as outlined in Chapter 12.

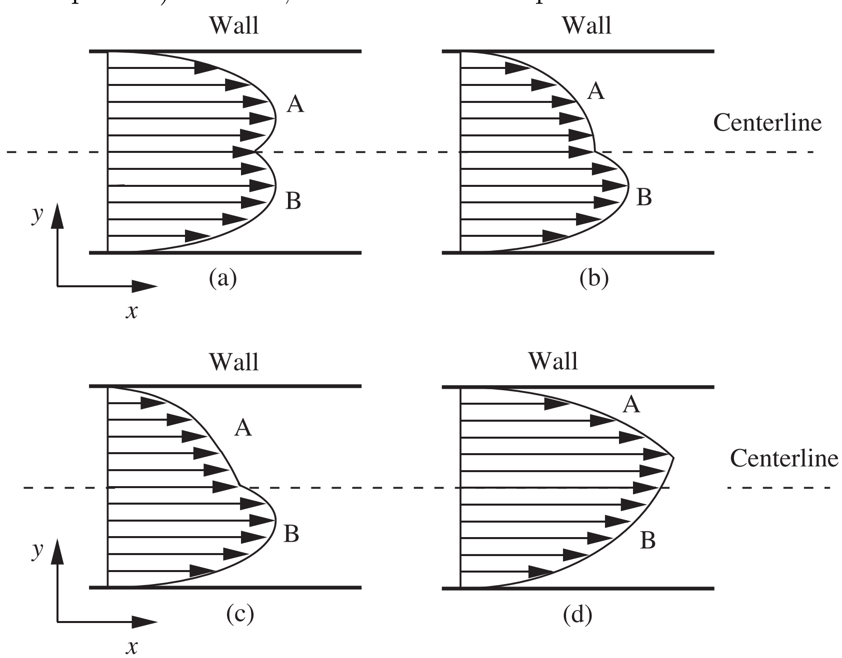


Fig. P6.11 Proposed velocity profiles for immiscible liquids.

11. *Permissible velocity profiles—E.* Consider the shear stress τ_{yx}; why must it be *continuous*—in the y direction, for example—and *not* undergo a sudden step-change in its value? Two *immiscible* Newtonian liquids A and B are in steady laminar flow between two parallel plates. Profile A meets the centerline normally in (b), but at an angle in (c); the maximum velocity in (d) does not coincide with the centerline. Which—if any—of the velocity profiles shown in Fig. P6.11 are impossible? Explain your answers carefully.

12. *"Creeping" flow past a sphere—D.* Figure P6.12 shows the steady, "creeping" (very slow) flow of a fluid of viscosity μ past a sphere of radius a. Far away from the sphere, the pressure is p_∞ and the undisturbed fluid velocity is U in the positive z direction. The following velocity components and pressure have been proposed in spherical coordinates:

$$\begin{aligned}
p &= p_\infty - \frac{3\mu U a}{2r^2}\cos\theta,\\
v_r &= U\left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right)\cos\theta,\\
v_\theta &= -U\left(1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right)\sin\theta,\\
v_\phi &= 0.
\end{aligned}$$

Assuming the velocities are sufficiently small so that terms such as $v_r(\partial v_r/\partial r)$ can be neglected, and that gravity is unimportant, prove that these equations do indeed satisfy the following conditions, and therefore are the solution to the problem:

(a) The continuity equation.
(b) The r and θ momentum balances.
(c) A pressure of p_∞ and a z velocity of U far away from the sphere.
(d) Zero velocity components on the surface of the sphere.

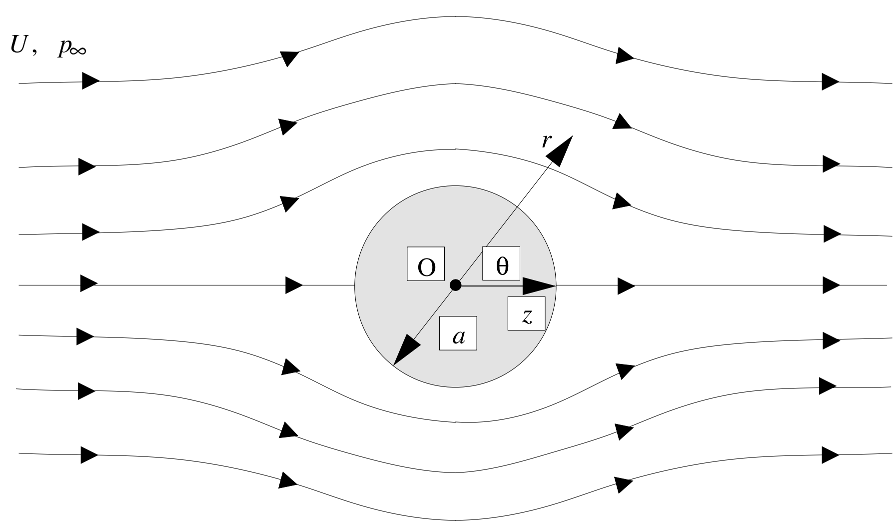


Fig. P6.12 Viscous flow past a sphere.

Also derive an expression for the net force exerted in the z direction by the fluid on the sphere, and compare it with that given by Stokes' law in Eqn. (4.11).

Note that the problem is one in *spherical* coordinates, in which the z axis has no formal place, except to serve as a reference direction from which the angle θ is measured. There is also symmetry about this axis, such that any derivatives in the ϕ direction are zero. *Note*: the actual *derivation* of these velocities, *starting* from the equations of motion, is fairly difficult!

13. *Torque in a Couette viscometer—M.* Fig. P6.13 shows the horizontal cross section of a concentric cylinder or "Couette" viscometer, which is an apparatus for determining the viscosity μ of the fluid that is placed between the two vertical cylinders. The inner and outer cylinders have radii of r_1 and r_2, respectively. If the inner cylinder is rotated with a steady angular velocity ω, and the outer cylinder is stationary, derive an expression for v_θ (the θ velocity component) as a function of radial location r.

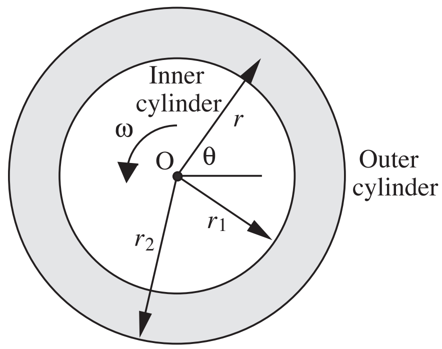


Fig. P6.13 Section of a Couette viscometer.

If, further, the torque required to rotate the *inner* cylinder is found to be T per unit length of the cylinder, derive an expression whereby the unknown viscosity μ can be determined, in terms of T, ω, r_1, and r_2. *Hint*: you will need to consider one of the shear stresses given in Table 5.8.

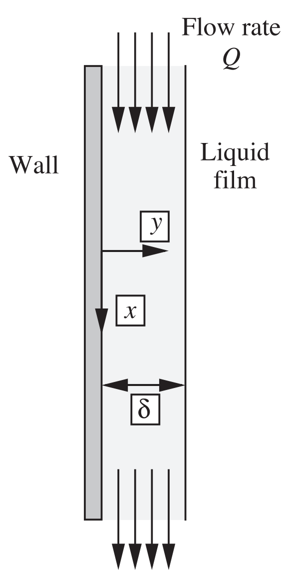


Fig. P6.14 Wetted-wall column.

14. *Wetted-wall column—M.* Fig. P6.14 shows a "wetted-wall" column, in which a thin film of a reacting liquid of viscosity μ flows steadily down a plane wall, possibly for a gas-absorption study. The volumetric flow rate of liquid is specified as Q per unit width of the wall (normal to the plane of the diagram).

Assume that there is only one nonzero velocity component, v_x, and that this does not vary in the x direction, and that the gas exerts negligible shear stress on the liquid film. Starting from the equations of motion, derive an expression for the "profile" of the velocity v_x (as a function of ρ, μ, g, y, and δ), and also for the film thickness, δ (as a function of ρ, μ, g, and Q).

15. *Simplified view of a Weissenberg rheogoniometer—M.* Consider the Weissenberg rheogoniometer with a very shallow cone; thus, referring to Fig. E6.7, $\beta = \pi/2 - \alpha$, where α is a small angle.

(a) Without going through the complicated analysis presented in Example 6.7, outline your reasons for supposing that the shear stress at any location on the cone is:

$$(\tau_{\theta\phi})_{\theta=\beta} = \frac{\omega\mu}{\alpha}.$$

(b) Hence, prove that the torque required to hold the cone stationary (or to rotate the lower plate) is:

$$T = \frac{2}{3}\frac{\pi\omega\mu R^4}{H}.$$

(c) By substituting $\beta = \pi/2 - \alpha$ into Eqn. (E6.7.13) and expanding the various functions in power series (only a very few terms are needed), prove that $g(\beta) = 1/(2\alpha)$, and that Eqn. (E6.7.18) again leads to the expression just obtained for the torque in part (b) above.

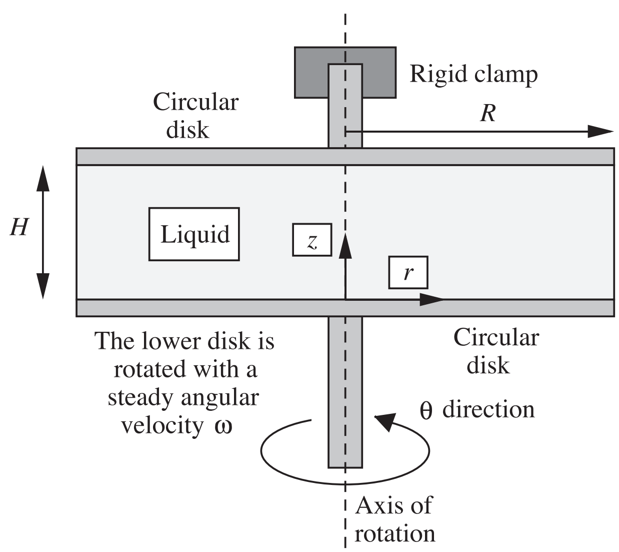


Fig. P6.16 Cross section of parallel-disk rheometer.

16. *Parallel-disk rheometer—M.* Fig. P6.16 shows the diametral cross section of a viscometer, which consists of two opposed circular horizontal disks, each of radius R, spaced by a vertical distance H; the intervening gap is filled by a liquid of constant viscosity μ and constant density. The upper disk is stationary, and the lower disk is rotated at a steady angular velocity ω in the θ direction.

There is only *one* nonzero velocity component, v_θ, so the liquid everywhere moves in circles. Simplify the general continuity equation in cylindrical coordinates, and hence deduce those coordinates (r?, θ?, z?) on which v_θ may depend.

Now consider the θ-momentum equation, and simplify it by eliminating all zero terms. Explain briefly: (a) why you would expect $\partial p/\partial\theta$ to be zero, and (b) why you *cannot* neglect the term $\partial^2 v_\theta/\partial z^2$.

Also explain briefly the logic of supposing that the velocity in the θ direction is of the form $v_\theta = r\omega f(z)$, where the function $f(z)$ is yet to be determined. Now substitute this into the simplified θ-momentum balance and determine $f(z)$, using the boundary conditions that v_θ is zero on the upper disk and $r\omega$ on the lower disk.

Why would you designate the shear stress exerted by the liquid on the lower disk as $\tau_{z\theta}$? Evaluate this stress as a function of radius.

17. *Screw extruder optimum angle—M.* Note that the flow rate through the die of Example 6.5, given in Eqn. (E6.5.10), can be expressed as:

$$Q = \frac{c(p_2 - p_3)}{\mu D},$$

in which c is a factor that accounts for the geometry.

Suppose that this die is now connected to the exit of the extruder studied in Example 6.4, and that $p_1 = p_3 = 0$, both pressures being atmospheric. Derive an expression for the optimum flight angle θ_{opt} that will *maximize* the flow rate Q_y through the extruder and die. Give your answer in terms of any or all of the constants c, D, h, L_0, r, W, μ, and ω.

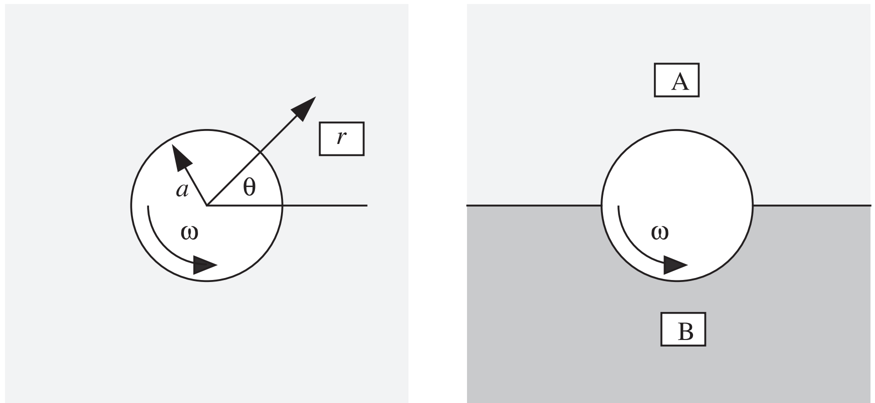
Under what conditions would the pressure at the exit of the extruder have its largest possible value p_{2max}? Derive an expression for p_{2max}.

18. *Annular flow in a die—E.* Referring to Example 6.5, concerning annular flow in a die, answer the following questions, giving your explanation in both cases:

(a) What form does the velocity profile, $v_z = v_z(r)$, assume as the radius r_1 of the inner cylinder becomes vanishingly small?

(b) Does the maximum velocity occur halfway between the inner and outer cylinders, or at some other location?

19. *Rotating rod in a fluid—M.* Fig. P6.19(a) shows a horizontal cross section of a long vertical cylinder of radius a that is rotated steadily counterclockwise with an angular velocity ω in a very large volume of liquid of viscosity μ. The liquid extends effectively to infinity, where it may be considered at rest. The axis of the cylinder coincides with the z axis.



(a) (b)

Fig. P6.19 Rotating cylinder in (a) a single liquid, and (b) two immiscible liquids.

(a) What type of flow is involved? What coordinate system is appropriate?

(b) Write down the differential equation of mass and that one of the three general momentum balances that is most applicable to the determination of the velocity v_θ.

(c) Clearly stating your assumptions, simplify the situation so that you obtain an ordinary differential equation with v_θ as the dependent variable and r as the independent variable.

(d) Integrate this differential equation, and introduce any boundary condition(s), and prove that $v_\theta = \omega a^2/r$.

(e) Derive an expression for the shear stress $\tau_{r\theta}$ at the surface of the cylinder. Carefully explain the plus or minus sign in this expression.

(f) Derive an expression that gives the torque T needed to rotate the cylinder, per unit length of the cylinder.

(g) Derive an expression for the vorticity component $(\nabla \times \mathbf{v})_z$. Comment on your result.

(h) Fig. P6.19(b) shows the initial condition of a mixing experiment in which the cylinder is in the middle of two immiscible liquids, A and B, of *identical* densities and viscosities. After the cylinder has made one complete rotation, draw a diagram that shows a representative location of the interface between A and B.

20. *Two-phase immiscible flow—M.* Fig. P6.20 shows an apparatus for measuring the pressure drop of two immiscible liquids as they flow horizontally between two parallel plates that extend indefinitely normal to the plane of the diagram. The liquids, A and B, have viscosities μ_A and μ_B, densities ρ_A and ρ_B, and volumetric flow rates Q_A and Q_B (per unit depth normal to the plane of the figure), respectively. Gravity may be considered unimportant, so that the pressure is essentially only a function of the horizontal distance, x.

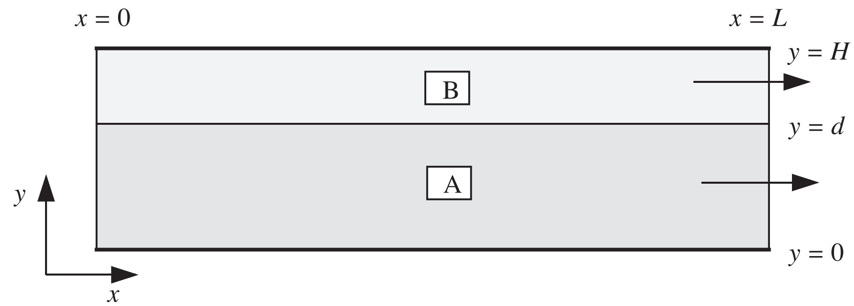


Fig. P6.20 Two-phase flow between parallel plates.

(a) What type of flow is involved?

(b) Considering layer A, start from the differential equations of mass and momentum, and, clearly stating your assumptions, simplify the situation so that you obtain a differential equation that relates the horizontal velocity v_{xA} to the vertical distance y.

(c) Integrate this differential equation so that you obtain v_{xA} in terms of y and any or all of d, dp/dx, μ_A, ρ_A, and (assuming the pressure gradient is uniform) two arbitrary constants of integration, say, c_{1A} and c_{2A}. Assume that a similar relationship holds for v_{xB}.

(d) Clearly state the four boundary and interfacial conditions, and hence derive expressions for the four constants, thus giving the velocity profiles in the two layers.

(e) Sketch the velocity profiles and the shear-stress distribution for τ_{yx} between the upper and lower plates.

(f) Until now, we have assumed that the interface level $y = d$ is known. In reality, however, it will depend on the relative flow rates Q_A and Q_B. Show clearly how this dependency could be obtained, but do not actually carry the calculations through to completion.

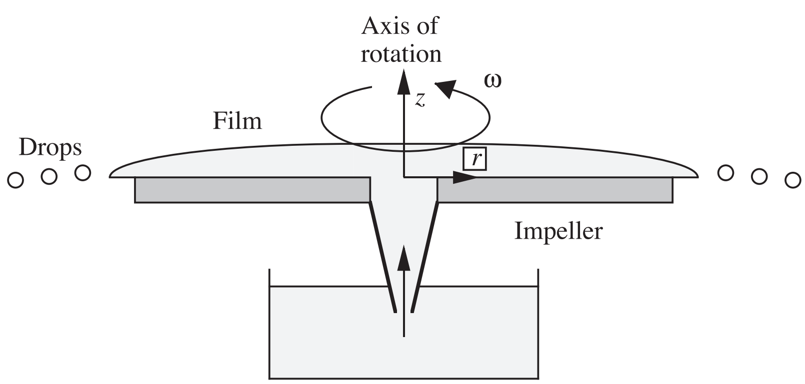


Fig. P6.21 Rotating-impeller humidifier.

21. *Room humidifier—M.* Fig. P6.21 shows a room humidifier, in which a circular impeller rotates about its axis with angular velocity ω. A conical extension dips into a water-bath, sucking up the liquid (of density ρ and viscosity μ), which then spreads out over the impeller as a thin laminar film that rotates everywhere with an angular velocity ω, eventually breaking into drops after leaving the periphery.

(a) Assuming incompressible steady flow, with symmetry about the vertical (z) axis, and a relatively small value of v_z, what can you say from the continuity equation about the term:

$$\frac{\partial(rv_r)}{\partial r}\,?$$

(b) If the pressure in the film is everywhere atmospheric, the only significant inertial term is v_θ^2/r, and information from (a) can be used to neglect one particular term, to what two terms does the r momentum balance simplify?

(c) Hence prove that the velocity v_r in the radial direction is a half-parabola in the z direction.

(d) Derive an expression for the total volumetric flow rate Q, and hence prove that the film thickness at any radial location is given by:

$$\delta = \left(\frac{3Q\nu}{2\pi r^2\omega^2}\right)^{1/3},$$

where ν is the kinematic viscosity.

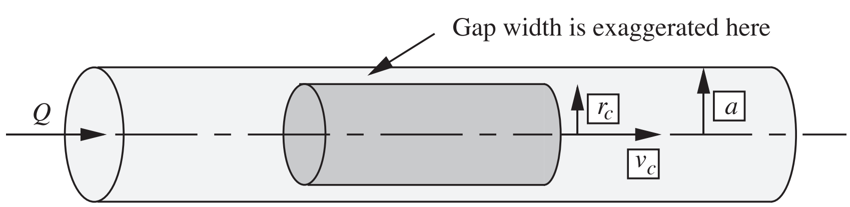


Fig. P6.22 Transport of inner cylinder.

22. *Transport of inner cylinder—M (C).* As shown in Fig. P6.22, a long solid cylinder of radius r_c and length L is being transported by a viscous liquid of the same density down a pipe of radius a, which is much smaller than L. The annular gap, of extent $a - r_c$, is much smaller than a. Assume: (a) the cylinder remains concentric within the pipe, (b) the flow in the annular gap is laminar, (c) the shear stress is essentially constant across the gap, and (d) entry and exit effects can be neglected. Prove that the velocity of the cylinder is given fairly accurately by:

$$v_c = \frac{2Q}{\pi(a^2 + r_c^2)},$$

where Q is the volumetric flow rate of the liquid upstream and downstream of the cylinder. *Hint*: concentrate first on understanding the physical situation. Don't rush headlong into a lengthy analysis with the Navier-Stokes equations!

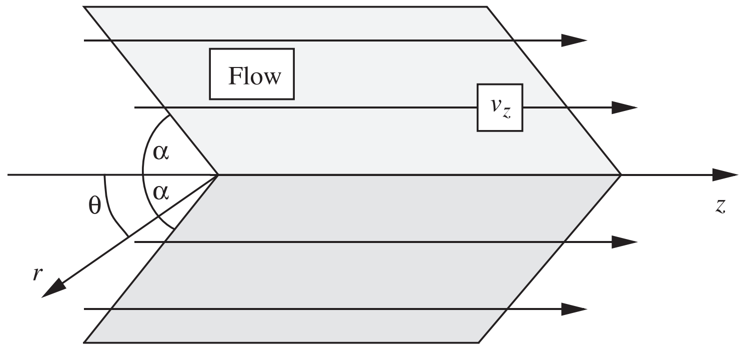


Fig. P6.23 Flow between inclined planes.

23. *Flow between inclined planes—M (C).* A viscous liquid flows between two infinite planes inclined at an angle 2α to each other. Prove that the liquid velocity, which is everywhere parallel to the line of intersection of the planes, is given by:

$$v_z = \frac{r^2}{4\mu}\left(\frac{\cos 2\theta}{\cos 2\alpha} - 1\right)\left(-\frac{\partial p}{\partial z}\right),$$

where z, r, θ are cylindrical coordinates. The z-axis is the line of intersection of the planes and the r-axis ($\theta = 0$) bisects the angle between the planes. Assume laminar flow, with:

$$\frac{\partial p}{\partial z} = \mu \nabla^2 v_z = \mu \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}\right],$$

and start by proposing a solution of the form $v_z = r^n g(\theta)(-\partial p/\partial z)/\mu$, where the exponent n and function $g(\theta)$ are to be determined.

24. *Immiscible flow inside a tube—D (C).* A film of liquid of viscosity μ_1 flows down the inside wall of a circular tube of radius $(\lambda + \Delta)$. The central core is occupied by a second immiscible liquid of viscosity μ_2, in which there is no net vertical flow. End effects may be neglected, and steady-state circulation in the core liquid has been reached. If the flow in both liquids is laminar, so that the velocity profiles are parabolic as shown in Fig. P6.24, prove that:

$$\alpha = \frac{2\mu_2\Delta^2 + \lambda\mu_1\Delta}{4\mu_2\Delta + \lambda\mu_1},$$

where Δ is the thickness of the liquid film, and α is the distance from the wall to the point of maximum velocity in the film. Fig. P6.24 suggests notation for solving the problem. To save time, assume parabolic velocity profiles without proof: $u = a + bx + cx^2$ and $v = d + ey + fy^2$.

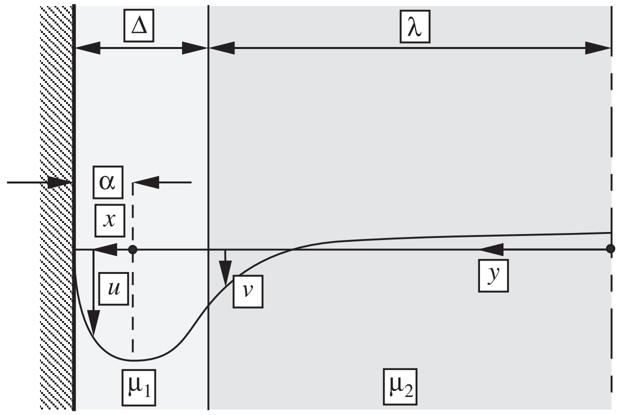


Fig. P6.24 Flow of two immiscible liquids in a pipe. The thickness Δ of the film next to the wall is exaggerated.

Discuss what happens when $\lambda/\Delta \to \infty$; and also when $\mu_2/\mu_1 \to 0$; and when $\mu_1/\mu_2 \to 0$.

25. *Blowing a polyethylene bubble—D.* For an incompressible fluid in cylindrical coordinates, write down:

(a) The continuity equation, and simplify it.

(b) Expressions for the viscous normal stresses σ_{rr} and $\sigma_{\theta\theta}$, in terms of pressure, viscosity, and strain rates.

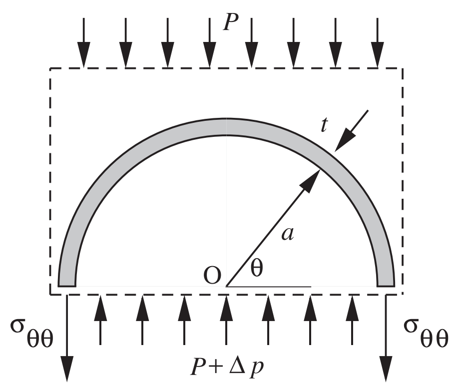


Fig. P6.25 Cross section of half of a cylindrical bubble.

A polyethylene sheet is made by inflating a cylindrical bubble of molten polymer effectively at constant length, a half cross section of which is shown in Fig. P6.25. The excess pressure inside the bubble is small compared with the external pressure P, so that $\Delta p \ll P$ and $\sigma_{rr} \doteq -P$.

By means of a suitable force balance on the indicated control volume, prove that the circumferential stress is given by $\sigma_{\theta\theta} = -P + a\Delta p/t$. Assume pseudo-steady state—that is, the circumferential stress just balances the excess pressure, neglecting any acceleration effects.

Hence, prove that the expansion velocity v_r of the bubble (at $r = a$) is given by:

$$v_r = \frac{a^2 \Delta p}{4\mu t},$$

and evaluate it for a bubble of radius 1.0 m and film thickness 1 mm when subjected to an internal gauge pressure of $\Delta p = 40$ N/m^2. The viscosity of polyethylene at the appropriate temperature is 10^5 N s/m^2.

26. *Surface-tension effect in spinning—M.* Example 6.6 ignored surface-tension effects, which would increase the pressure in a filament of radius R approximately by an amount σ/R, where σ is the surface tension. Compare this quantity with the reduction of pressure, $\mu dv_z/dz$, caused by viscous effects, for a polymer with $\mu = 10^4$ P, $\sigma = 0.030$ kg/s^2, $L = 1$ m, R_L (exit radius) $= 0.0002$ m, $v_{z0} = 0.02$ m/s, and $v_{zL} = 2$ m/s. Consider conditions both at the beginning and end of the filament. Comment briefly on your findings.

27. *Radial pressure variations in spinning—M.* Example 6.6 assumed that the variation of pressure across the filament was negligible. Investigate the validity of this assumption by starting with the suitably simplified momentum balance:

$$\frac{\partial p}{\partial r} = \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial (r v_r)}{\partial r} \right) + \frac{\partial^2 v_r}{\partial z^2} \right].$$

If v_z is the local axial velocity, prove that the corresponding increase of pressure from just inside the free surface (p_R) to the centerline (p_0) is:

$$p_0 - p_R = \frac{\mu v_z (\ln \beta)^3 R^2}{4 L^3},$$

where $\beta = v_{zL}/v_{z0}$. Obtain an expression for the ratio $\xi = (p_R - p_0)/(\mu dv_z/dz)$, in which the denominator is the pressure decrease due to viscosity when crossing the interface from the air into the filament, and which *was* accounted for in Example 6.6.

Estimate ξ at the beginning of the filament for the situation in which $\mu = 10^4$ P, $L = 1$ m, R_L (exit radius) $= 0.0002$ m, $v_{z0} = 0.02$ m/s, and $v_{zL} = 2$ m/s. Comment briefly on your findings.

28. *Condenser with varying viscosity—M.* In a condenser, a viscous liquid flows steadily under gravity as a uniform laminar film down a vertical cooled flat plate. Due to conduction, the liquid temperature T varies linearly across the film, from T_0 at the cooled plate to T_1 at the hotter liquid/vapor interface, according to $T = T_0 + y(T_1 - T_0)/h$. The viscosity of the liquid is given approximately by $\mu = \mu^*(1 - \alpha T)$, where μ^* and α are constants.

Prove that the viscosity at any location can be reexpressed as:

$$\mu = \mu_0 + cy, \qquad \text{in which} \qquad c = \frac{\mu_1 - \mu_0}{h},$$

where μ_0 and μ_1 are the viscosities at temperatures T_0 and T_1, respectively.

What is the expression for the shear stress τ_{yx} for a liquid for steady flow in the x-direction? Derive an expression for the velocity v_x as a function of y. Make sure that you *do not* base your answer on any equations that assume *constant* viscosity.

Sketch the velocity profile for both a small and a large value of α.

29. *True/false.* Check *true* or *false*, as appropriate:

(a) For horizontal flow of a liquid in a rectangular duct between parallel plates, the pressure varies linearly both in the direction of flow and in the direction normal to the plates. T ☐ F ☐

(b) For horizontal flow of a liquid in a rectangular duct between parallel plates, the boundary conditions can be taken as zero velocity at one of the plates and *either* zero velocity at the other plate *or* zero velocity gradient at the centerline. T ☐ F ☐

(c) For horizontal flow of a liquid in a rectangular duct between parallel plates, the shear stress varies from zero at the plates to a maximum at the centerline. T ☐ F ☐

(d) For horizontal flow of a liquid in a rectangular duct between parallel plates, a measurement of the pressure gradient enables the shear-stress distribution to be found. T ☐ F ☐

(e) In fluid mechanics, when integrating a partial differential equation, you get one or more *constants* of integration, whose values can be determined from the boundary condition(s). T ☐ F ☐

(f) For flows occurring between $r = 0$ and $r = a$ in cylindrical coordinates, the term $\ln r$ may appear in the final expression for one of the velocity components. T ☐ F ☐

(g) For flows in ducts and pipes, the volumetric flow rate can be obtained by differentiating the velocity profile. T ☐ F ☐

(h) Natural convection is a situation whose analysis depends on *not* taking the density as constant everywhere. T ☐ F ☐

(i) A key feature of the Weissenberg rheogoniometer is the fact that a conical upper surface results in a uniform velocity gradient between the cone and the plate, for all values of radial distance. T ☐ F ☐

(j) If, in three dimensions, the pressure obeys the equation $\partial p/\partial y = -\rho g$, and both $\partial p/\partial x$ and $\partial p/\partial z$ are nonzero, then integration of this equation gives the pressure as $p = -\rho g y + c$, where c is a constant. T ☐ F ☐

(k) If two immiscible liquids A and B are flowing in the x direction between two parallel plates, both the velocity v_x and the shear stress τ_{yx} are continuous at the interface between A and B, where the coordinate y is normal to the plates. T ☐ F ☐

(l) In compression molding of a disk between two plates, the force required to squeeze the plates together decreases as time increases. T ☐ F ☐

(m) For flow in a wetted-wall column, the pressure increases from atmospheric pressure at the gas/liquid interface to a maximum at the wall. T ☐ F ☐

(n) For one-dimensional flow in a pipe—either laminar or turbulent—the shear stress τ_{rz} varies linearly from zero at the wall to a maximum at the centerline. T ☐ F ☐

(o) In Example 6.1, for flow between two parallel plates, the shear stress τ_{yx} is negative in the upper half (where $y > 0$), meaning that physically it acts in the opposite direction to that indicated by the convention. T ☐ F ☐